ELASTIC WAVE PROPAGATION IN A CONTINUOUS MEDIUM

Herein, we will deal mainly with the humble wave equation,

$$\ddot{\vec{u}} \propto \nabla^2 \vec{u}$$

, where *u* is the displacement, and ∇^2 is the Laplacian operator.

Laplacian = " $\nabla \cdot \nabla$ "= "del dot product div"

$$\begin{aligned}\nabla^2 u &= u_{,ii} \\ &= \nabla \bullet \left(\frac{\partial\, u(x_1,x_2,x_3)}{\partial x_1}, \frac{\partial\, u(x_1,x_2,x_3)}{\partial x_2}, \frac{\partial\, u(x_1,x_2,x_3)}{\partial x_3} \right) \\ &= \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}, \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}, \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\end{aligned}$$

A conceptual, physical interpretation of this equation can be that the acceleration experienced by the passing of a wave is a function of the difference between the local and average surrounding displacements (Laplacian).

In other words, the spatial derivatives of the strain (which is a spatial derivative of the displacement) are proportional to the stress the material is experiencing.

The stress that the material experiences constant of proportionality depends on the speed of the wave through the material (see the potential expressions at the end of the chapter)

An elastic wave is a deformation of the body that travels throughout the body in all directions. We can examine the deformation over a period of time by fixing our look on just one point in space. This is the case of fixing geophones or seismometers in the field, or Eulerian description.

We will begin by a simple case, assuming that we have (1) an isotropic medium, that is, the elastic properties or wave velocity, are not directionally dependent and that (2) our medium is continuous. By examining a balance of forces across an elemental volume and relating the forces on the volume to an ideal elastic response of the volume using Hooke's Law we will derive one form of the elastic wave equation.

Let us begin by examining the balance of forces and mass (Newton's Second Law) for a very small elemental volume. The effect of traction forces and additional body forces ($\vec{f}$) is to generate an acceleration ($\ddot{\vec{u}}$) per unit volume of mass or density (ρ):

$$\rho\, \ddot{u}_i = \sigma_{ij,j} + f_i, \qquad (1)$$

->To Acoustic Wave Equation

where the double-dot above u, denotes the second partial derivative with respect to time

$$(\frac{\partial^2 u_i}{\partial t^2}).$$

(Note that a dimensional analysis confirms this statement)

The deformation in the body is achieved by displacing individual particles about their central resting point. Because we consider that the behavior is essentially elastic, the particles will eventually come to rest at their original starting point. Displacement for each point in space is described by a vector with a tail at that point.

$$\vec{u} = (u_1, u_2, u_3)$$

Each component of the displacement, $\boldsymbol{u}_i$ depends on the location within the body and at what stage of the wave propagation we are considering.

Density (ρ) is a scalar property that depends on what point in 3-D space we consider:

$$\rho = \rho(x, x_2, x_3) \text{ or, in other words}$$

$$\rho(x_1, x_2, x_3) = \rho(\vec{x})$$

Body forces such as the effect of gravity are discarded. The **homogeneous equation for motion** (1) states that the acceleration a particle of rock undergoes while under the influence of traction forces is proportional to the <u>stress changes across its volume:</u>

(for $j=1,2,3$)

Each basis vector component of the acceleration, as for example $i=1$, is expressed as

$$\ddot{u}_1 \hat{x}_1 = \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \right) \hat{x}_1$$

Finally, in complete indicial notation: $\ddot{\boldsymbol{u}}_i = \boldsymbol{\sigma}_{ij,j}$, for a given elemental volume.

Remember from the chapter on strain that the infinitesimal deformation at each point depends on the <u>gradients in the displacement</u> field:

$$e_{pq} = \frac{1}{2}(u_{p,q} + u_{q,p})$$

$$= \frac{1}{2}(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}) \quad \text{(substituting, } p \text{ for } i \text{ and } q \text{ for } j\text{)}$$

Empirically, it has been shown that for small strains (e.g., 10^{-5}), and over short periods of time (e.g., Lay and Wallace, 1995) rocks behave as ideal elastic solids. The most general form of Hooke's Law for an ideal elastic solid is:

$$\boldsymbol{\sigma}_{ij} = c_{ijpq} \boldsymbol{e}_{pq} \qquad (4)$$

where c_{ijpq} is a **fourth-order** tensor (3 dimensions) containing de 3^4=**81** elastic constants or matrix components that define the elastic properties of the material in the an anisotropic and inhomogeneous medium. Each component c_{ijpq} or elastic constant has dimensions of **pressure**. Each component c_{ijpq} is independent of the strain e_{ij} and for this reason is called a 'constant' although elastic constants vary throughout space as a function of position.

We can reduce the number of constants to **two** in various steps. First, we can reduce the **number to 36** because it follows that since σ_{ij} y e_{ij} are symmetric:

$$c_{jipq} = c_{ijpq} \text{ and } c_{ijqp} = c_{ijpq}.$$

That is, adjacent the first two indices can be interchanged and the last two indices can be interchanged as well.

Through thermodynamic considerations (Green, 1838,1839; in Pujol, 2013) it was demonstrated mathematically and eventually through experimental support of the original arguments that

$$c_{pqij} = c_{ijpq}$$

That is adjacent pairs of subindices can be interchanged

so that even in the case of anisotropy the number of constants can be reduced to **21** (Cauchy and Poisson had estimated them to be **15** at one time (Pujol, 2013)) However, it is possible to often solve many geological problems by considering that rocks have isotropic elastic properties. The assumption of isotropy reduces the number of independent elastic constants to just **2.** In summary for an **isotropic**, **continuous medium** we can reduce the elastic constant tensor to the following:

$$c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \quad (5)$$

where λ y μ are known as the Lamé elastic parameters or properties. Lamé parameters λ y μ can be expressed in terms of other familiar elastic parameters such as Young's modulus E and Poisson's ratio σ:

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}; \mu = \frac{E}{2(1+\sigma)}$$

Other elastic parameters can also be expressed in terms of λ y μ (also known as Lamé's first and second parameters. For example, incompressibility K relates the change in pressure surrounding a body to the corresponding relative change in volume of the body:

$$K = -V\frac{\Delta P}{\Delta V} = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\sigma)} \quad (7)$$

Substitution of equation (5) into equation (4) shows that traction forces and strain are related for an isotropic medium in the following manner:

$$\sigma_{ij} = \left[\lambda\delta_{ij}\delta_{pq} + \mu\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right)\right]e_{pq}$$

$$= \lambda\delta_{ij}\delta_{pq}e_{pq} + \mu\delta_{ip}\delta_{jq}e_{pq} + \mu\delta_{iq}\delta_{jp}e_{pq}$$

If we add over repeated subindices:

$$= \lambda \delta_{ij} \left(\delta_{11} e_{11} + \delta_{22} e_{22} + \delta_{33} e_{33} \right)$$
$$+ \mu \delta_{i(p=1,2,3)} \delta_{jq} \left(e_{1q} + e_{2q} + e_{3q} \right)$$
$$+ \mu \delta_{iq} \delta_{j(p=1,2,3)} \left(e_{1q} + e_{2q} + e_{3q} \right)$$

From the definition of a Kronecker delta, the only terms that will be non-zero and contribute to the stress tensor will be those which have their subindices equal to each other. That is for the second term on the right of the equals sign, values exist if $p = i$ and $q = j$. Similarly, for the third term on the right of the equals sign values exist if $p = j$ and $q = i$. With this simplification we arrive at:

$$= \lambda \delta_{ij} e_{kk} + \mu \delta_{ii} \delta_{jj} e_{ij} + \mu \delta_{ii} \delta_{jj} e_{ji}$$

Because the deformation tensor is symmetric $e_{ij} = e_{ji}$ leading to the result that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \qquad (8a)$$

->To Acoustic Wave Equation

In experiments we observe displacement, ground velocity and acceleration so it makes sense to express the stresses in terms displacements,

$$\sigma_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$ or, in complete indicial notation:

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \qquad (8b)$$

since

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} \right) \text{ and } e_{kk} = e_{11} + e_{22} + e_{33} = u_{k,k} = \nabla \cdot \vec{u}$$

Note too, that $\nabla \cdot \vec{u} = \frac{\Delta V}{V}$, (where ΔV is the relative change in volume, for infinitesimal deformations)

We obtain the wave equation for displacements in a general isotropic medium by substituting (8b) into the equation of motion

$$\rho \ddot{\vec{u}}_i = \sigma_{ij,j} + f_i \quad (1)$$

$$= \left[\lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \right]_{,j} + f_i$$

$$= (\lambda \delta_{ij} u_{k,k})_{,j} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij}) + f_i ,$$

$$= \lambda_{,j}\delta_{ij}u_{k,k} + \lambda\delta_{ij}u_{k,k,j} + \mu_{,j}(u_{i,j} + u_{j,i}) + \mu(u_{i,jj} + u_{j,ij}) + f_i$$ after expansion using the product rule.

Let us take each of the terms on the right hand side separately to demonstrate the application of indicial notation. For each term I, only the case where j=i can contribute in the Kronecker delta, so

$$\begin{aligned} \lambda_{,i}\delta_{ij}u_{k,k} &= \lambda_{,i}\delta_{ii}u_{k,k} \\ &= \lambda_{,i}u_{k,k} \\ &\equiv \lambda_{,i}u_{j,j} \end{aligned}$$

because we can interchange the repeated k's by repeated j's because they both signify summation over the range of values for j; i.e., 1 through 3.

$$= \underbrace{\lambda_{,i}u_{j,j}}_{I} + \underbrace{\lambda u_{k,k,i}}_{II} + \underbrace{(+\mu)u_{j,ij}}_{III} + \underbrace{\mu u_{i,jj}}_{IV} + + \underbrace{\mu_{,j}(u_{i,j} + u_{j,i})}_{V} + f_i \quad (9)$$

After some algebra we show that an alternative expression can be obtained by adding (9) vectorially from $i = 1,2,3$ to arrive at:

$$\rho\ddot{\vec{u}} = (\lambda + \mu)\nabla(\nabla \bullet \vec{u}) + \mu\nabla^2\vec{u} + \nabla\lambda\nabla \bullet \vec{u} + \nabla\mu \times \nabla \times \vec{u} + 2(\nabla\mu \bullet \nabla)\vec{u} + \vec{f} \quad (10)$$

Two fundamental body wave types: P waves and S waves

From the equation of motion (10) in vectorial form, we can demonstrate (Poisson, 18..), that in an **infinite**, **elastic**, and **isotropic**, **homogeneous** medium two types of particle motion associated with traveling trains of deformation can be predicted.

Since λ and μ are constant in a homogeneous medium, we have that $\nabla\lambda$ and $\nabla\mu$ both equal zero because there are no spatial changes in their values. This leaves:

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \bullet \vec{u}) + \mu\nabla^2\vec{u}$$

But, we can use the identity:

$$\mu(\nabla \times \nabla \times \vec{u}) = \mu\nabla(\nabla \bullet \vec{u}) - (\nabla^2\vec{u})\mu$$, (identity 1) so that

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \bullet \vec{u}) - \mu(\nabla \times \nabla \times \vec{u}) \quad (11)$$

Through the Helmholtz theorem (aka fundamental theorem of vector calculus), the displacement vector field can be decomposed into two independent component fields: a scalar field together with a vectorial field.

$$\vec{u} = \nabla\Theta + \nabla \times \vec{\Omega}$$

The first term on the right (scalar) is non-rotational and the second term (a vectorial field) is rotational. The scalar field of displacement can not experience rotations and the vectorial field can not experience a divergence. In other words

$$\nabla \bullet (\nabla \times "vector") \equiv 0 \qquad \text{(identity 2)}$$

$$\nabla \times \nabla("scalar") \equiv 0, \qquad \text{(identity 3)}$$

Now, if we take the divergence of (11) we can simplify the whole expression. The second term on the right becomes zero because $\nabla \times \vec{u}$ is a vector quantity, and its rotational is zero (identity 2):

$$\nabla\left(\rho \frac{\partial^2 \vec{u}}{\partial t^2}\right) = \nabla\left[(\lambda + 2\mu)\nabla(\nabla \bullet \vec{u})\right]$$

$$\rho \frac{\partial^2 (\nabla \bullet \vec{u})}{\partial t^2} = \nabla^2 (\nabla \bullet \vec{u})(\lambda + 2\mu)$$

We can change variable names by defining a new scalar field variable $\Theta = \nabla \bullet \vec{u}$ so that the immediately preceding expression looks like, (a.k.a.) the scalar wave equation:

$$\rho \frac{\partial^2 \Theta}{\partial t^2} = \nabla^2 \Theta (\lambda + 2\mu)$$

$$\frac{\rho}{\lambda + 2\mu} \ddot{\Theta} = \nabla^2 \Theta, \text{ where } \ddot{\Theta} = \frac{\partial^2 \Theta}{\partial t^2}$$ (similar to Ikelle and Amundsen's Eq. 2.163)

In order to propagate this type of deformation through the medium the body must expand and contract (divergence is non-zero). This portion of the displacement field can not have a rotation component; only a divergence component.

One solution to this scalar wave equation is to make

$$\Theta = V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

$$\nabla^2 \Theta = V_P^2 = \frac{\lambda + 2\mu}{\rho}$$

Note that the integral of the potential is also the integral of the velocity or simply the displacement.

Now, if we take the rotational of the general equation of motion (11) i.e.,

$$\nabla \times \left(\rho \ddot{\vec{u}}\right) = \nabla \times \left[(\lambda + 2\mu)\nabla\left(\nabla \bullet \vec{u}\right) - \mu\left(\nabla \times \nabla \times \vec{u}\right)\right]$$

$$\rho \frac{\partial^2 \left(\nabla \times \vec{u}\right)}{\partial t^2} = (\lambda + 2\mu)\nabla \times \nabla(\nabla \bullet \vec{u}) - \mu \nabla \times \nabla \times \left(\nabla \times \vec{u}\right)$$

Because $\nabla \bullet \vec{u}$ is a scalar field and the rotational of the gradient of this field is zero (identity 3). So, the first term on the right of the equation goes to zero:

$$\rho \frac{\partial^2 \left(\nabla \times \vec{u}\right)}{\partial t^2} = -\mu \nabla \times \nabla \times \left(\nabla \times \vec{u}\right)$$

We can now change variable names by defining a new vector field variable $\vec{\Omega} = \nabla \times \vec{u}$ so that the immediately preceding expression looks like:

$$\begin{aligned}
\rho \frac{\partial^2 \Omega}{\partial t^2} &= -\mu \nabla \times \left(\nabla \times \vec{\Omega}\right) \\
&= -\mu \left\{\nabla\left(\nabla \bullet \left[\left(\nabla \times \vec{\Omega}\right)\right]\right)\right\} + \nabla^2 \vec{\Omega} \\
&= -\frac{\mu}{\rho} \vec{\Omega}
\end{aligned}$$

(similar to Ikelle and Amundsen (2005) Eq. 2.164)

because the first term goes to zero since the divergence of the rotational of $\vec{\Omega}$ is zero (identity 2).

One solution to this vector wave equation is to make

$$\vec{\Omega} = V_S = \sqrt{\frac{\mu}{\rho}}$$

The separation into different wavefields is useful to know for practical reasons. If a full 3-component, 2-D array of geophones can record a wavefield, then the wavefield can be broken out by separating the field into the transverse waves and the compressional waves. The divergence of the full 3-C, 2-D array data set would leave behind only transverse waves. The curl of the same data set would leave behind only compressional waves. Of course, the wavefield can not be separated out completely because our 3D experiments collect data over a 2-D grid so that the partial derivatives in the z direction may have to be considered as constant.

In Summary, we have seen that

(1) the strain is related to displacement field by the following relationship

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

(2) and that Newton's second law provides the following relationship between particle acceleration and stress:

$$\ddot{\boldsymbol{u}}_i = \sigma_{ij,j}$$

(3) Also, Hooke's Law can be written as:

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

(4) Finally, when we substitute Hooke's Law into Newton's Law through the expression of stress as a function of displacements we can eventually find the solution to two wave equations that give us the speed of propagation of two waves through elastic, isotropic material:

$$\vec{\Omega} = V_S = \sqrt{\frac{\mu}{\rho}} \text{ and } \Theta = V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

Appendix for section on the wave equation ->Back to text

In this section we state that vectorial manipulation of the expression:

$$\rho\ddot{\vec{u}} = \underbrace{(\lambda+\mu)u_{j,ij}}_{I} + \underbrace{\mu u_{i,jj}}_{II} + \underbrace{\lambda_{,i}u_{j,j}}_{III} + \underbrace{\mu_{,j}(u_{i,j}+u_{j,i})}_{IV} + f_i \qquad (9)$$

for $i = 1,2,3$ leads to the alternative expression:

$$\rho\ddot{\vec{u}} = (\lambda+\mu)\nabla(\nabla \bullet \vec{u}) + \mu\nabla^2\vec{u} + \nabla\lambda\nabla \bullet \vec{u} + \nabla\mu \times \nabla \times \vec{u} + 2(\nabla\mu \bullet \nabla)\vec{u} + \vec{f}$$

In order to show the steps in detail, let us examine each of the terms I through IV on the right hand side of equation (9).

Starting with I: e.g., when $k=1,2,3,$ for commutative partial derivatives

$$\begin{aligned} u_{k,ki} &= \frac{\partial^2 u_1}{\partial x_1 \partial x_i} + \frac{\partial^2 u_2}{\partial x_2 \partial x_i} + \frac{\partial^2 u_3}{\partial x_3 \partial x_i} \\ &= \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_i}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_i}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_i}\right) \\ &= u_{k,ik} \equiv u_{j,ij} \end{aligned}$$

For II:

$$\begin{aligned} u_{i,jj} &= \frac{\partial^2 u_i}{\partial x_1 \partial x_1} + \frac{\partial^2 u_i}{\partial x_2 \partial x_2} + \frac{\partial^2 u_i}{\partial x_3 \partial x_3} \\ &= \nabla^2(u_i) \end{aligned}$$

For III

For IV