Voigt Notation for Reducing the Dimension of a Tensor

In matrix form, we can use also use the Voigt notation to write out the non-diagonalized isotropic stiffness tensor where the stress and strain can be represented as a six-component vector rather than a nine-element square matrix:

Additional simplification of the stress-strain can be realized through simplifying the matrix notation for stresses and strains. We can replace the indices as follows:

$$11 \to 1 \qquad 23 \to 4$$
$$22 \to 2 \qquad 13 \to 5$$
$$33 \to 3 \qquad 12 \to 6$$

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} \to \begin{pmatrix} 1 & 6 & 5 \\ 6 & 2 & 4 \\ 5 & 4 & 3 \end{pmatrix}$$

Voigt Notation

Isotropic stress tensor

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \quad (1.1)$$

where, $\varepsilon_I \Leftrightarrow \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$ and, $e_{ij} = \begin{pmatrix} \varepsilon_1 & \frac{1}{2}\varepsilon_6 & \frac{1}{2}\varepsilon_5 \\ \frac{1}{2}\varepsilon_6 & \varepsilon_2 & \frac{1}{2}\varepsilon_4 \\ \frac{1}{2}\varepsilon_5 & \frac{1}{2}\varepsilon_4 & \varepsilon_3 \end{pmatrix}$

Recall that: $e_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$

Anisotropic stress tensor

If we assume the anisotropy to be weak in a medium then we can approximate the stiffness matrix as the following. As most sediments consist of clays, the symmetry of clays is a good first approximation to the seismic symmetry seen in nature, or the Transverse Isotropy with a vertical axis of symmetry, or **VTI** (Tsvankin, 2012)

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{11}-2c_{66} & 0 & 0 & 0 & 0 \\ c_{11}-2c_{66} & c_{13} & 0 & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Voigt notation helps us handle the 2^{nd} order stress and strain tensors even when we have 6 elastic constants (note that one is a linear combination of the other two) as is with the case of anisotropy