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“...it is necessary to have three components of translation, six components of strain, and three components of rotation to fully characterize the change in the medium around point x ” (Suriyanto, 2006), but in practice we do not consider the rotational component (Aki and Richards, 2002).

INTRODUCTION

In the most general case, we can consider continuous deformation to be the result of a translation, rotation and stretching along the major axes. In the classical derivation of seismic-elastic strain we do not consider the rotational components. Seismic deformation results by subjecting a rock to stress with a minimal amount of temporary deformation. Because we are considering the elastic wave field we will consider deformations that too small to reach the yield strength of the rock and change shape or size permanently. It is helpful to describe a general deformation field as a function of position in three-dimensional space:

$$\vec{u} = \mathbf{u}(x_1, x_2, x_3)$$

We choose to examine the deformational behavior of a very small portion of rock say about a few mineral grains, at a point

$$\vec{P} = P(x_1, x_2, x_3).$$

We refer to deformation at this point by $\vec{u}(\vec{P})$. By reducing the size of our problem to short distances we are less likely to err in approximating a general function of deformation by one that is

only linear. This is analogous to expanding a function using a Taylor series expansion but only using the first two terms of the expansion.

One-dimensional strain

In the one-dimensional case, deformation, $u(x_1)$, can be calculated in the vicinity of the origin O (where, $x_1 = 0$, i.e. Taylor Series) as a function of the distance away from the origin:

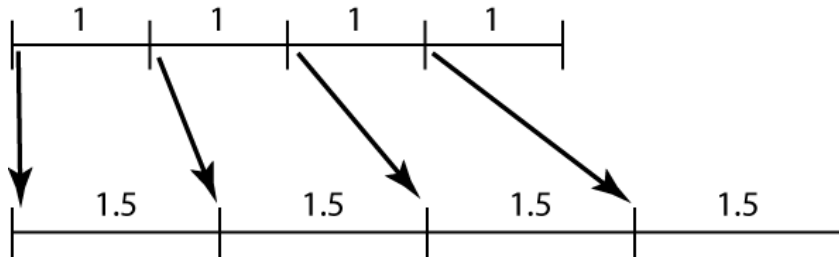
$$u(x_1)|_o = u(O) + x_1 \frac{du(x_1 - O)}{dx_1} + \frac{1}{2!} (x_1 - O)^2 \frac{d^2u(x_1 - O)}{dx_1^2} + \dots$$

Substituting for a specific value of $x_1 = a$, we have

$$u(a) = u(0) + x_1 \frac{du(a - 0)}{dx_1}$$

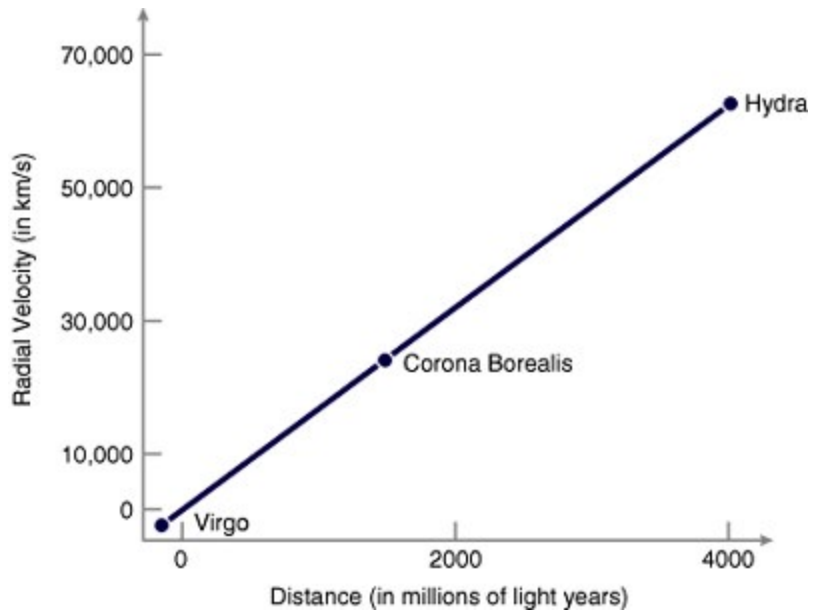
if we only consider the first term of the expansion.

If the higher-order terms are assumed insignificant, the deformation can be described as a simple linear function. The deformation at any point along the x_1 axis is a function of a small deformation at the origin plus an additional value proportional to distance from the origin.



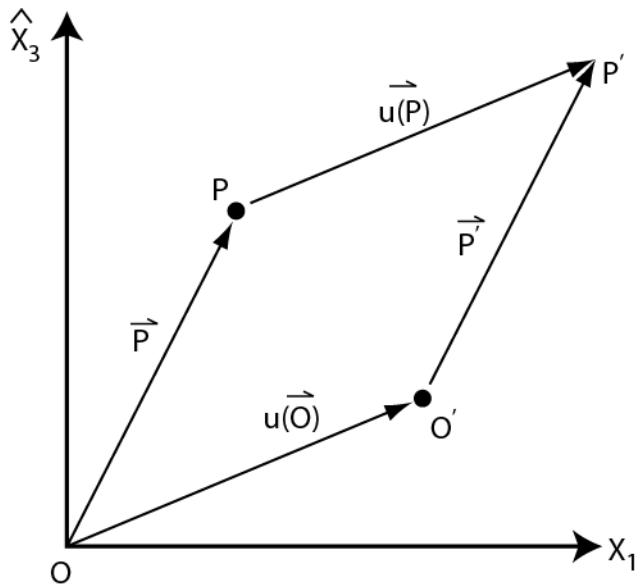
Curiously, there is a connection between homogeneous, linear strain (above) and the expansion of the universe. If we assume that the universe is expanding at the same throughout (homogeneous) and that it is proceeding at a constant rate, this is equivalent to saying that strain is homogeneous.

In a 1-D sense while the rate of deformation is homogeneous, the accumulated effect of the strain as seen from the origin is greater at greater distances from the observer where the cumulative velocity of the expansion is linear with offset between the observer and the target (sensor, in our case). The constant rate of linear deformation of the universe is known as the Hubble constant. The 2011 Nobel Prize in physics was awarded to those (Saul Perlmutter, Brian P. Schmidt, Adam G. Riess) who discovered that the rate of expansion of the universe is actually increasing and not constant.



(http://ircamera.as.arizona.edu/astr_250/Lectures/Lec_23sml.htm)

Two-dimensional strain



Two-dimensional case ($i=1,2; j=1,2$)

Three-dimensional strain

Note that I will intentionally use the symbol “ Δ ” in order to remark on its difference, if only temporarily, from the gradient of a scalar field that we have already seen. This symbol Δ does not represent the gradient of a scalar field, but the gradient tensor of the displacement vector field.

Compare the simple gradient of a scalar field:

$$\nabla(u) = \frac{\partial u}{\partial x_1} \hat{x}_1 + \frac{\partial u}{\partial x_2} \hat{x}_2 + \frac{\partial u}{\partial x_3} \hat{x}_3$$

to the gradient tensor of the displacement field:

$$\Delta \bar{u}(\bar{P}) = \left. \frac{\partial \bar{u}(\bar{P})}{\partial x_1} \right|_{x_2, x_3} x_1 \hat{x}_1 + \left. \frac{\partial \bar{u}(\bar{P})}{\partial x_2} \right|_{x_1, x_3} x_2 \hat{x}_2 + \left. \frac{\partial \bar{u}(\bar{P})}{\partial x_3} \right|_{x_1, x_2} x_3 \hat{x}_3, \text{ where}$$

$$\Delta \bar{u}_1(\bar{P}) = \left(\frac{\partial \bar{u}_1(\bar{P})}{\partial x_1} x_1 + \frac{\partial \bar{u}_1(\bar{P})}{\partial x_2} x_2 + \frac{\partial \bar{u}_1(\bar{P})}{\partial x_3} x_3 \right) \hat{x}_1$$

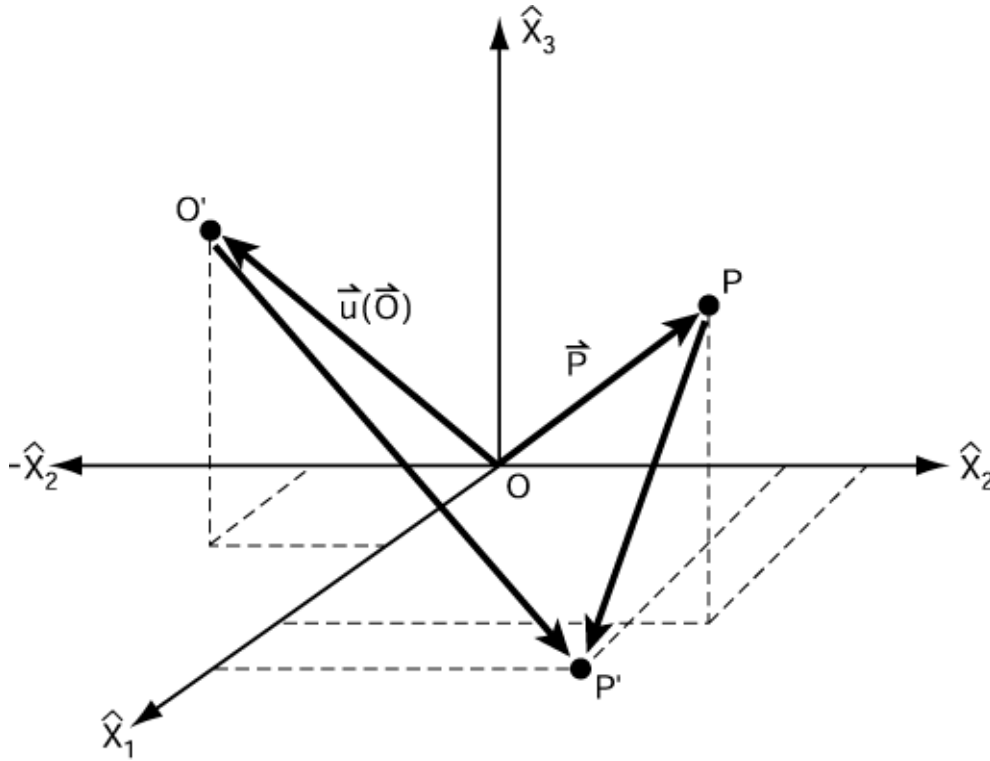
$$\Delta \bar{u}_2(\bar{P}) = \left(\frac{\partial \bar{u}_2(\bar{P})}{\partial x_1} x_1 + \frac{\partial \bar{u}_2(\bar{P})}{\partial x_2} x_2 + \frac{\partial \bar{u}_2(\bar{P})}{\partial x_3} x_3 \right) \hat{x}_2 \quad (2)$$

$$\Delta \bar{u}_3(\bar{P}) = \left(\frac{\partial \bar{u}_3(\bar{P})}{\partial x_1} x_1 + \frac{\partial \bar{u}_3(\bar{P})}{\partial x_2} x_2 + \frac{\partial \bar{u}_3(\bar{P})}{\partial x_3} x_3 \right) \hat{x}_3$$

All the components of the displacement contribute to the gradient of the deformation,

where $u_2(\bar{P})$ means the second component of the displacement vector for point P. This can be expressed as:

$$\begin{pmatrix} \Delta \bar{u}_1(\bar{P}) \\ \Delta \bar{u}_2(\bar{P}) \\ \Delta \bar{u}_3(\bar{P}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{u}_1(\bar{P})}{\partial x_1} & \frac{\partial \bar{u}_1(\bar{P})}{\partial x_2} & \frac{\partial \bar{u}_1(\bar{P})}{\partial x_3} \\ \frac{\partial \bar{u}_2(\bar{P})}{\partial x_1} & \frac{\partial \bar{u}_2(\bar{P})}{\partial x_2} & \frac{\partial \bar{u}_2(\bar{P})}{\partial x_3} \\ \frac{\partial \bar{u}_3(\bar{P})}{\partial x_1} & \frac{\partial \bar{u}_3(\bar{P})}{\partial x_2} & \frac{\partial \bar{u}_3(\bar{P})}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



For the 3-D strain shown in the last figure above, there is a Taylor series expansion for functions with three variables. In three-dimensions, by analogy to the one-dimensional conditions, the deformation function can be approximated by the first two terms of the expansion in Taylor series:

$$\bar{u}(\bar{P}) = \bar{u}(\bar{O}) + \Delta\bar{u}(\bar{P}),$$

where $\Delta\bar{u}$ is here defined as the discrete differential of the function, valid for the very small volumes of rock we consider but whose limit for a continuous medium, is the differential du . Remember that \bar{u} (vector) describes the behavior of deformation in all 3-dimensions at once. The total deformation will have contributions from all three directions. We can examine contributions to deformation from each of these three directions

$$\Delta\bar{u}(\bar{P}) = \left. \frac{\partial\bar{u}(\bar{P})}{\partial x_1} \right|_{x_2, x_3} x_1 \hat{x}_1 + \left. \frac{\partial\bar{u}(\bar{P})}{\partial x_2} \right|_{x_1, x_3} x_2 \hat{x}_2 + \left. \frac{\partial\bar{u}(\bar{P})}{\partial x_3} \right|_{x_1, x_2} x_3 \hat{x}_3$$

For example term $\left. \frac{\partial\bar{u}(\bar{P})}{\partial x_1} \right|_{x_2, x_3}$ is read as “the partial differential of the function \bar{u} with respect to the \hat{x}_1 basis vector direction”. The vertical bar with the sub-indices signifies that during

differentiation any of the other two components of \vec{u} in either the \hat{x}_2 or \hat{x}_3 directions are constant and differentiate to zero. Note that because \vec{u} is itself a function of a three-variable function of position, u_1 in the \hat{x}_1 direction **can** have terms that relate to the x_2 and x_3 value of \vec{P} .

In this figure above, we see that there are two types of three-dimensional vectors used to describe the deformation. One vector relates position of any point in the undeformed state $\vec{P} = (x_1, x_2, x_3)$ with respect to the origin in the **undeformed** state (\vec{O}). Another vector relates the new position of the point $\vec{P}' = (x'_1, x'_2, x'_3)$ **after** the deformation, with respect to the new origin (\vec{O}'). The difference in the deformation between that which occurs at the origin (\vec{O}) and at \vec{P} is the deformation at \vec{P} relative to the new deformed coordinate system.

The discrete differential at P is a three-dimensional vector, with the following components:

$$\Delta\vec{u}(\vec{P}) = (\Delta\bar{u}_1(\vec{P}), \Delta\bar{u}_2(\vec{P}), \Delta\bar{u}_3(\vec{P})),$$

where $\vec{u}(\vec{P}) = (\bar{u}_1(\vec{P}), \bar{u}_2(\vec{P}), \bar{u}_3(\vec{P}))$

is the general deformation function that we approximate in a linear fashion.

After a coordinate transformation, $\vec{O} \rightarrow \vec{O}'$ and $\vec{P} \rightarrow \vec{P}'$
Three-dimensional case ($i=1,2,3; j=1,2,3$)

\vec{P}' is the vector that describes the deformation with respect to the new origin \vec{O}' , that is we have experienced a coordinate transformation.

O is related to O' through the displacement vector

$$\vec{u}(\vec{O}) = O',$$

and P is related to P' through the displacement vector:

$$\vec{u}(\vec{P}) = \vec{u}(\vec{O}) + \Delta\vec{u}(\vec{P}), \quad (1)$$

similar to the Taylor expansion for the one-dimensional case.

$\Delta\vec{u}$ is the spatial change of the displacement vector. In the most general case, each component of the displacement vector is a function of three other components:

$$\bar{u} = f(x_1, x_2, x_3)$$

If we want to express the deformation in the deformed co-ordinate system we can use (1), and the illustration to determine that:

$$\begin{aligned}\bar{P}' - \bar{u}(\bar{P}) &= \bar{P} \\ \bar{P}' - \bar{u}(\bar{O}) - \Delta\bar{u}(\bar{P}) &= \bar{P}, \\ \bar{P}' &= \bar{P} + \Delta\bar{u}(\bar{P}) + \bar{u}(\bar{O})\end{aligned}$$

We can express the displacements as a gradient tensor where each displacement component has contributions from the other three dimensions (see box):

By assuming that P is a vector function (!) and in indicial notation,

$$\Delta u_i = \frac{\partial u_j}{\partial x_i} x_j$$

From (1) and (2) we have that the deformed point located with respect to the origin in the deformed system is:

$$\begin{aligned}\bar{P}'_1 = x'_1 &= x_1 + \Delta u_1 \\ &= x_1 + \frac{\partial u_1}{\partial x_1} x_1 + \frac{\partial u_1}{\partial x_2} x_2 + \frac{\partial u_1}{\partial x_3} x_3\end{aligned}$$

$$\begin{aligned}\bar{P}'_2 = x'_2 &= x_2 + \Delta u_2 \\ &= x_2 + \frac{\partial u_2}{\partial x_1} x_1 + \frac{\partial u_2}{\partial x_2} x_2 + \frac{\partial u_2}{\partial x_3} x_3\end{aligned}$$

etc.,

Using indicial notation we can rewrite the above expressions as:

$$\bar{P}'_i = x'_i = x_i + \frac{\partial u_i}{\partial x_j} x_j,$$

Using matrix notation we can also write this as:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (3)$$

Conditions for homogeneity in two-dimensions

We can now revert to two-dimensions to convey the same concept but with less notation than the three-dimensional case;

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4)$$

Deformation is **homogeneous** throughout a space when the following two conditions are met:

- (1) Parallel material lines in the undeformed space remain parallel following the deformation and
- (2) if straight lines in the undeformed space remain as straight lines following the deformation.

These two conditions are equivalent mathematically to stating that

$$\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}$$

(or $\frac{\partial u_i}{\partial x_j}$ in indicial notation),

does not change through space and can be considered to be of **constant value**: say, a, b, c, d . That is, we can write (4) as:

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5)$$

$$\text{where } a = 1 + \frac{\partial u_1}{\partial x_1}, \quad b = \frac{\partial u_1}{\partial x_2},$$

$$c = \frac{\partial u_2}{\partial x_1}, \text{ and } d = 1 + \frac{\partial u_2}{\partial x_2}$$

Actually, the assumption of homogeneous strain is scale-dependent. For a sufficiently small scale we can always consider the deformation to be homogeneous. Beyond the scale of homogeneity, we must start to deal with inhomogeneous deformation, also known as non-homogeneous or heterogeneous deformation. In order to deal with the inhomogeneity we would expand the equation (2) to include more terms. However, when dealing with elasticity, such as the propagation of waves, normally we have that the deformations are very small, a few percent at most and

$$\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}$$

$$\left(\text{or } \frac{\partial u_i}{\partial x_j} \text{ in indicial notation}\right) \ll 1.$$

In geological applications these quantities can be quite large and the values for a, b, c, d are not infinitesimal.

An example of deformation of a line

We can now demonstrate the condition of homogeneity by showing that a straight line remains straight throughout the deformation. Consider a line prior to deformation:

$$x_2 = mx_1 + k \quad (6)$$

From (5) and (6) we have that

$$x'_1 = ax_1 + bx_2 = ax_1 + b(mx_1 + k)$$

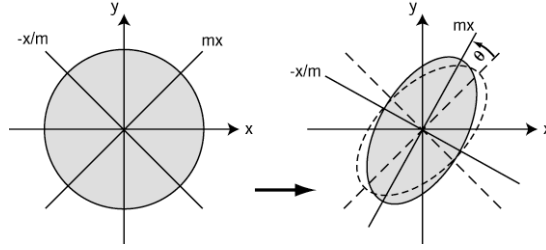
$$x'_2 = cx_1 + dx_2 = cx_1 + d(mx_1 + k)$$

By replacing the value of x_1 in the second equation with the value of x_1 taken from the first equation $x_1 = \frac{x'_1 - bk}{a + bm}$ we have:

$$\begin{aligned} x'_2 &= c \left(\frac{x'_1 - bk}{a + bm} \right) + dm \left(\frac{x'_1 - bk}{a + bm} \right) + dk \\ &= c \frac{x'_1}{a + bm} + dm \frac{x'_1}{a + bm} + k \left(\frac{-bc}{a + bm} \right) + k \left(\frac{-b dm}{a + bm} \right) + kd \left(\frac{a + bm}{a + bm} \right) \\ &= \frac{c + dm}{a + bm} x'_1 + k \left(\frac{-bc - bd + ad + bd}{a + bm} \right) \\ &= \left(\frac{c + dm}{a + bm} \right) x'_1 + \left(\frac{ad - bc}{a + bm} \right) k \\ &= m' + k' \end{aligned} \quad (7)$$

We have shown that the deformed straight line is also another line! Because m' does not depend on k , two parallel lines (same m' but a different k) will remain parallel after the deformation. They will have the same slope m' . We can also demonstrate that a circle will transform into an ellipse and that one ellipse will transform into another ellipse. A series of incremental homogeneous transformations can be described by a single finite-deformation ellipse.

(TBD: Conditions for principal strain axes: The case of general transformation: extension and contraction along two perpendicular axes plus a rotation)



Let us consider the homogeneous deformation:

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (8) \text{ or,}$$

$\mathbf{A}' = \mathbf{T}\mathbf{A}$, where

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad (9)$$

\mathbf{T} is matrix for deformation via a rotation transformation, \mathbf{A} is the matrix of another deformation and \mathbf{A}' is the resultant general deformation matrix. Note that matrix \mathbf{A} is symmetric so that it can be diagonalized. In this deformation, two perpendicular lines exist that remain perpendicular after the deformation in the same direction as before the deformation (eigenvectors). Along these directions there is either only extension or only shortening. These directions are the principal directions of deformation along which the deformation is either maximum or minimum.

We have that :

$$\begin{aligned} A &= a \cos \theta + b \sin \theta \\ B &= b \cos \theta + d \sin \theta \\ C &= -a \sin \theta + b \cos \theta \\ D &= -b \sin \theta + d \cos \theta \end{aligned}$$

As a result, the rotation angle θ is related to the general deformation components by:

$$\tan \theta = \frac{B - C}{A + D} \quad (9')$$

Once we arrive at a value for θ , we can multiply the left-hand side of equations for A and C by $\sin \theta$ and $\cos \theta$ to obtain

$$\begin{aligned} a &= A \cos \theta - C \sin \theta \\ b &= A \sin \theta + C \cos \theta \end{aligned}$$

We also multiply the left hand side of equations for B and D by $\sin \theta$ and $\cos \theta$, respectively to get:

$$d = B \sin \theta + D \cos \theta \quad (9'')$$

In decomposing \mathbf{A} from equation (9) the four components are shown to be trigonometric functions.

We could now just find the eigenvectors and show they are at right angles to each other (See exercises)! However, an alternative approach is to show that if two perpendicular lines exist in the deformed system they are also the principal deformation axes.

Let's start by considering two perpendicular lines:

$$x_2 = mx_1, \quad x_2 = -\frac{1}{m}x_1 \quad (10)$$

After application of the deformation matrix as given by:

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (11)$$

we have after substitution by (10)

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bmx_1 \\ bx_1 + dmx_1 \end{bmatrix} \text{ and } \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} ax_1 - \frac{bx_1}{m} \\ bx_1 - \frac{dx_1}{m} \end{bmatrix} \text{ which solve as:}$$

$$x'_2 = \frac{md + b}{mb + a} x'_1 \text{ and } x'_2 = \frac{mb - d}{ma - b} x'_1, \text{ respectively. We want to know whether}$$

$$m \equiv \frac{md + b}{mb + a} \text{ and } -\frac{1}{m} \equiv \frac{mb - d}{ma - b} ?$$

These conditions can be shown as true by substituting into the identity so that:

$$m + \left(-\frac{1}{m}\right)^{-1} = 0.$$

From the equation above we obtain that the deformed lines are also at right angles if the following combination is met:

$$m^2b + m(a - d) - b = 0 \quad (13)$$

TODO: Graphically, this can be expressed as:

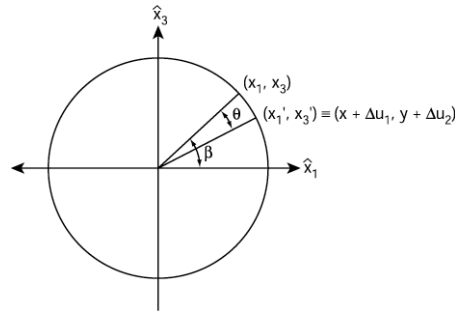
Question: Show that the homogeneous deformation of a circle:

$$x^2 + y^2 = 4$$

is an ellipse and draw the result.

Infinitesimal deformations

We now want to demonstrate that the strain theory we have looked at so far for finite geological deformations can also be applied to the very small deformations that are used in wave propagation theory and seismology.



Form equation (9) we have;

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \text{ where}$$

$$A = 1 + \frac{\partial u_1}{\partial x_1}, \quad B = \frac{\partial u_1}{\partial x_2},$$

$$C = \frac{\partial u_2}{\partial x_1}, \text{ and } D = 1 + \frac{\partial u_2}{\partial x_2}$$

Because deformations are very small, recall that $\frac{\partial u_i}{\partial x_j} \ll 1$ and that higher order derivatives will also be negligible. From equation (9') we have now:

$$\tan \theta = \frac{\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}}{2 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}} = \frac{\frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right)}{1 + \frac{1}{2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right)} \cong \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right)$$

As well,

$$\tan \theta \cong \sin \theta \cong \theta \text{ and } \cos \theta \cong 1 \quad (14a)$$

and to a first order equation (9'') becomes

$$\begin{aligned} a &\cong A - C\theta \\ b &\cong A\theta + C \\ d &\cong B\theta + D \end{aligned} \quad (14b)$$

For $\Delta \bar{u}(\bar{P}) = \bar{P}' - \bar{P}$, in the two-dimensional case we have:

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (15)$$

and in the infinitesimal case, using approximations (13) and (14):

$$\begin{aligned} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} &= \left\{ \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{bmatrix} A - C\theta & A\theta + C \\ A\theta + C & B\theta + D \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \left\{ \begin{bmatrix} A - 1 - C\theta & A\theta + C \\ A\theta + C & D - 1 + B\theta \end{bmatrix} + \theta \begin{bmatrix} A\theta + C & B\theta + D \\ c\theta - A & -A\theta - C \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (16)$$

If we disregard those second-order terms we get:

$$A - 1 - C\theta \rightarrow \frac{\partial u_1}{\partial x_1}$$

$$D - 1 + B\theta \rightarrow \frac{\partial u_2}{\partial x_2}$$

$$A\theta + C = \left(1 + \frac{\partial u_1}{\partial x_1} \right) \theta + \frac{\partial u_2}{\partial x_1} = \theta + \frac{\partial u_1}{\partial x_1} \theta + \frac{\partial u_2}{\partial x_1} \rightarrow \theta + \frac{\partial u_2}{\partial x_1}$$

Similarly:

$$\begin{aligned}\theta(A\theta^2 + C\theta) &\rightarrow 0 \\ \theta(B\theta^2 + D) &= B\theta^2 + \theta D \rightarrow \theta D = \left(1 + \frac{\partial u_2}{\partial x_2}\right)\theta \rightarrow \theta \\ C\theta - A &\rightarrow -\theta\end{aligned}$$

Then:

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \left\{ \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} + \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \end{bmatrix} \text{ or,}$$

if we use equation 13 and 14(a):

$$(17) \quad \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \left\{ \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} + \theta \begin{bmatrix} 0 & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \\ -\frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The first term on the right hand of the equation corresponds to the infinitesimal internal deformation of the rock body whereas the second term corresponds to an infinitesimal clockwise rotation by angle θ .

$$\begin{aligned}\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} &= \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} \theta x_2 \\ -\theta x_1 \end{bmatrix}\end{aligned}$$

Finally we arrive at the equation for the infinitesimal strain tensor e :

$$\begin{aligned} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} &= \left\{ \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} + \theta \begin{bmatrix} 0 & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \\ -\frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} \end{aligned}$$

In indicial notation we can write the infinitesimal strain tensor as:

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \text{ or for the general case: } e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$