FIELDS AND THE SYMMETRY OF PHYSICAL LAWS

Contents

1. DIFFERENT PHYSICAL MODELS FOR THE EARTH

In seismology we are primarily interested in the use of the wave equation at a sophisticated level. When we say the elastic wave field we describe the elastic behavior of earth materials to the passing of a wave by how fast they oscillate (Hz) and how much they move (amplitude in m) and the speed or acceleration with which they mover.

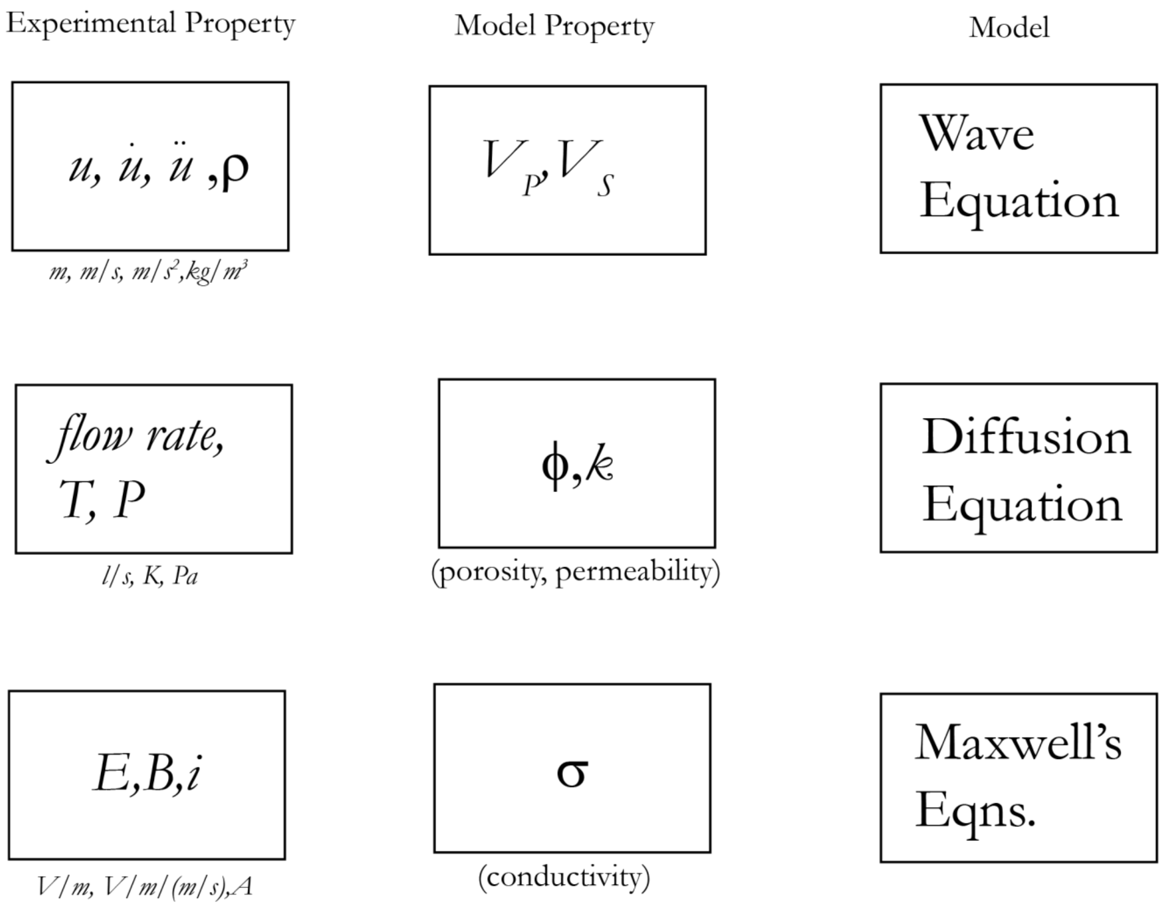
Although we are going to emphasize the mechanical properties of the earch in this course, 1uite often seismology alone is insufficient to understand a physical process in the earth and other physical models for the earth are required.

We talk of the electromagnetic field as a description of electrical and magnetic <u>properties (measurements)</u> within a given space. Electromagnetic models employ Maxwell's equations and give us a simplified view (model) of the earth in terms of some of these measured properties such as electrical conductivity.

A third manner of observing the world is to model the behavior of moving fluids through its connected pores. For this objective we employ the diffusion equation.

Other views of the earth exist based on how heat flows, how the lithosphere bends, and based on the distribution of the density of matter.

We talk of the gravitational field to describe gravity values and the direction of this force throughout a given body in space and time.



(adapted from Hadsell and Wiley, 1995)

1.2 Homogeneity, Isotropy

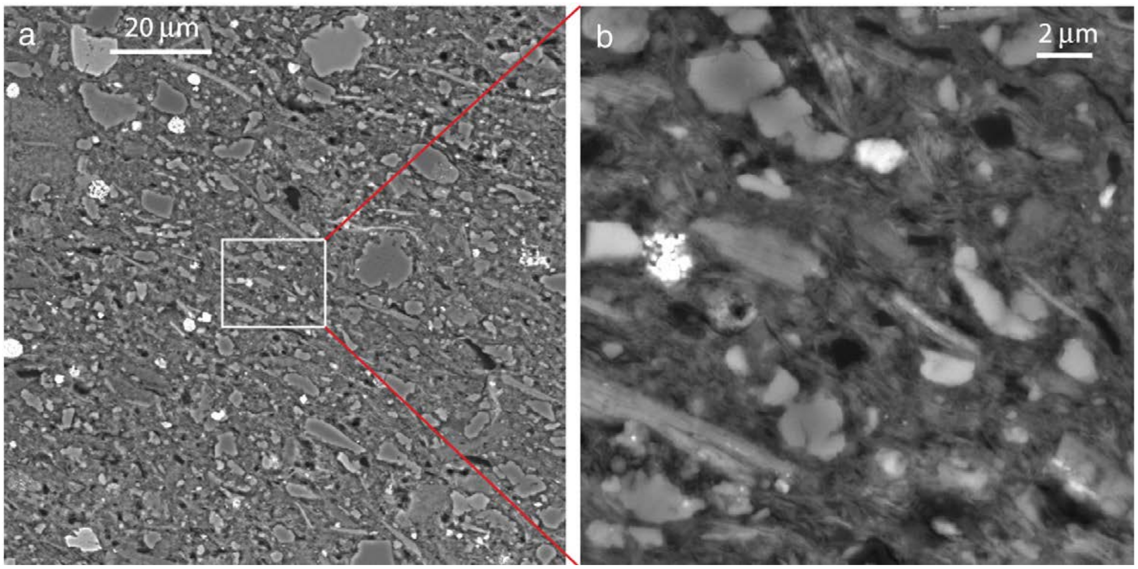
The field can also be described as isotropic when the properties at a location do not perceptibly change with the direction we consider.

The field can be considered non-homogeneous/heterogeneous when the property changes with either the spatial or temporal coordinate.

In the earth, vertically, the composition changes with depth, so we say that the earth is heterogeneous in a 1D sense.

In the earth, if the composition changes with depth and horizontally we say that the heterogeneity is two dimensional.

In reference to changes through time we also add a "dimension" in the description of the heterogeneity. I guess that someone could refer to 3D heterogeneity as having two spatial dimensions and a third temporal dimension instead of the usual 4D reference that is normally read.



(Josh et al., 2012)

Shale, as the above SEM picture may appear more heterogeneous at a higher resolution. The heterogeneity that systematically aligns grains in bands may set up an anisotropy at some given scale.

2. FIELDS: SCALARS, VECTORS AND TENSORS

Field: Is a physical quantifiable **property** that can be defined over some n-dimensional space

SCALAR FIELDS: Density field, temperature field, salinity field, Poisson's ratio field, shear modulus field, Young's modulus field

VECTOR FIELDS: Velocity field, Heat flow field, diffusivity of sediment field, gravity field, displacement, TENSOR FIELDS: Stress, strain, thermal conductivity

2.1 Scalar and vector properties

Each different field of physical properties has a different complexity that can described with increasingly complex, and more general, mathematics. If a property varies as a function ONLY of its position in space, i.e.

$$= f(x1, x2,x3),$$

then the property and field is known as scalar.

Vectors are quantities that have a directional property as well as a value in space. With a vector we know HOW MUCH it is worth and whether this quantity acts in a certain direction. A vector is described using three numbers.

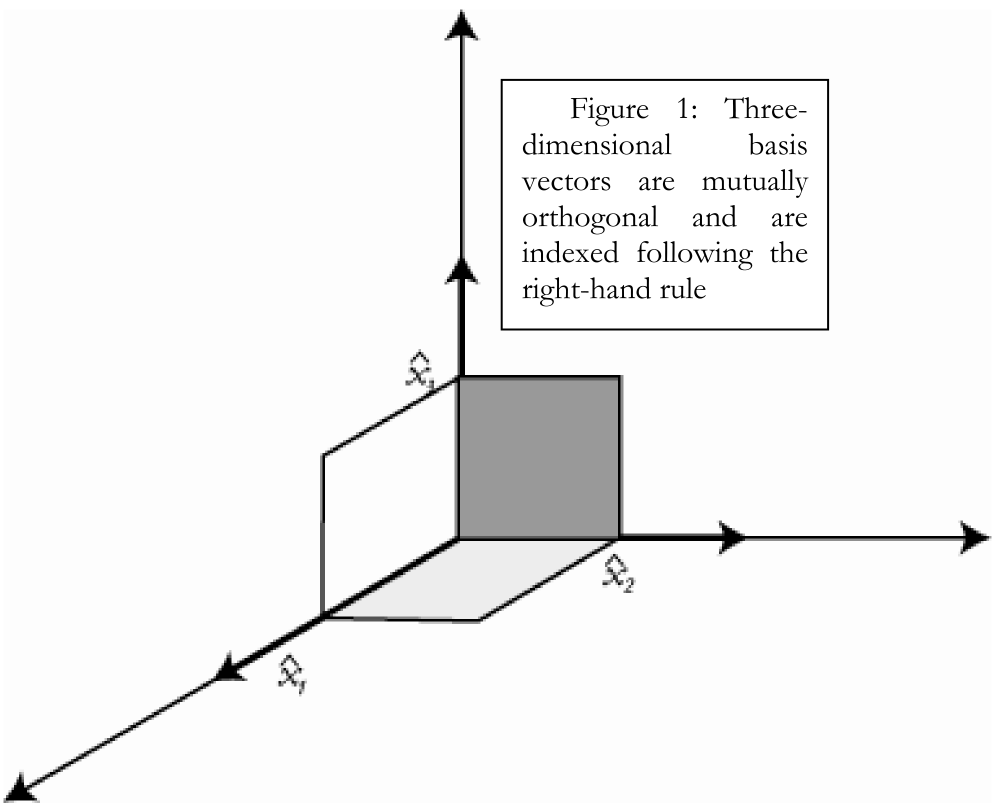


Figure 1: Three-dimensional basis vectors are mutually orthogonal and are indexed following the right-hand rule

Basis or unitary vectors in a cartesian co-ordinate system are mutually orthogonal (orthonormal) to each other, are of unit length and obey the right-hand rule. We can describe these vectors in several ways:

$$\vec{V} = a_1\hat{x}_1 + a_2\hat{x}_2 + a_3\hat{x}_3$$

or

$$\vec{V} = a_i\hat{x}_i$$

(If indices are repeated by convention we sum over them)

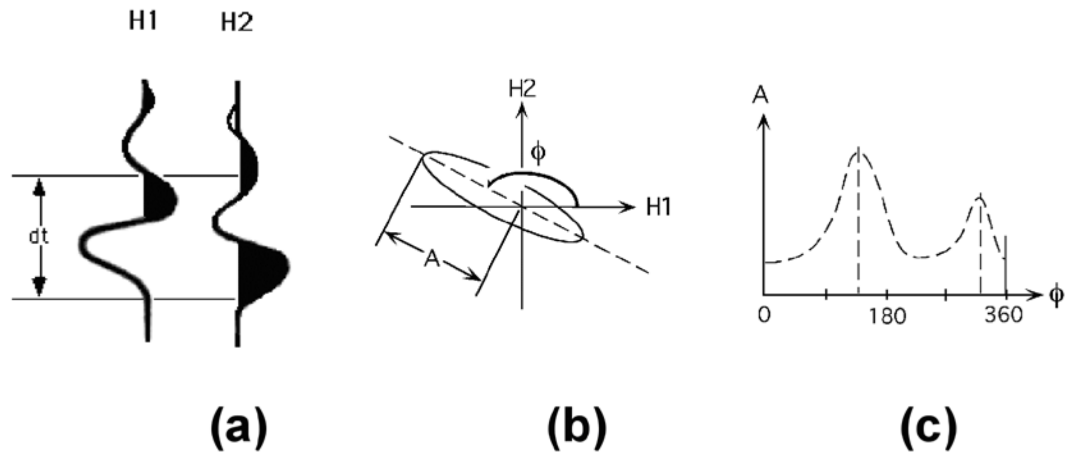
2.2 Invariance of Vectors under Linear Transformation

A vector property is also a property that is symmetric. That is, it does not matter whether these three numbers are different as measured with respect to different origins or frames of reference. These different numbers will still describe the same physical behavior, of say, the wave field.

Although it may seem obvious, certain physical laws do not change even if we modify our co-ordinate systems. That is, if we have our origin in one place and observe a wave field but then we choose to describe the wave field from a different origin or a different fixed frame of reference the observation will be the same. This wave field is symmetric.

Symmetry: (paraphrase: Hermann Weyl: a thing is symmetrical when after undergoing mathematical operations it looks the same as when we started, e.g. rotation of a plain undecorated vase by 180 degrees)

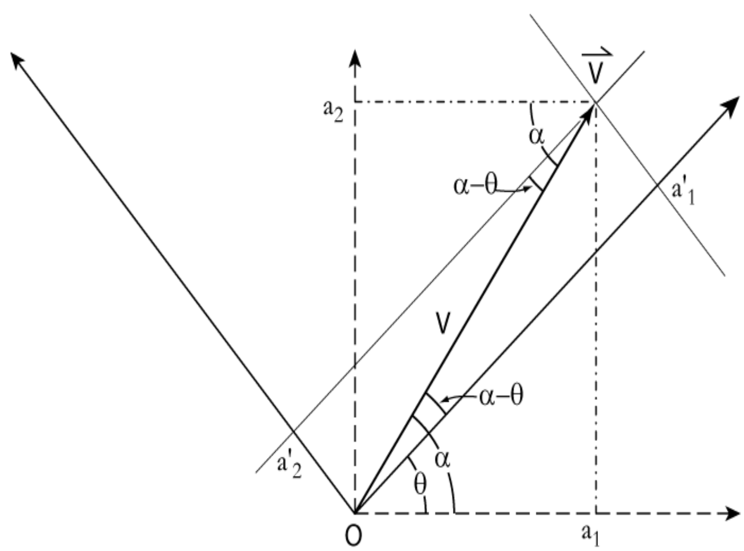
Example 1: In hydraulic fracturing, one manner for determining the direction of the originating microseism is to rotate the coordinate system until the direction in which particle motion of a particular wave mode is maximized, also known as the 'hodogram method' . In (Maxwell et al., 2010).



Guevara and Stewart, 1998.

Example 2: For example, if a vector is describing the velocity of a wave field at the earth's surface in a direction that is not perpendicular to the earth's surface we may choose to more conveniently rotate the co-ordinate reference frame in line with the particle motion. This happens in cases when studying anisotropy where we sometimes try out different rotations of the reference system until we maximize the wave energy coming from a particular direction.

Counter-clockwise rotation of a Cartesian Coordinate System (Left-handed system)



$$\begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix} = \begin{bmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$ or, more briefly expressed as

$\mathbf{V'} = \mathbf{TV}$, *->Proof of Invariance, using Indicial Notation*

where $\mathbf{V}'$ is the vector after the transformation expressed in components in terms of the rotated co-ordinate basis vectors and $\mathbf{V}$ is the vector before the transformation expressed in terms of components of the unaffected basis vector system.

Let us show how that the above is indeed true:

Before the co-ordinate rotation, the co-ordinates for $\mathbf{V}$ are:

$$a_1 = |\vec{V}|\cos\alpha, \text{ and } a_2 = |\vec{V}|\,sin\,\alpha$$

After the co-ordinate rotation the co-ordinates for $\mathbf{V}'$ are:

$$a_1' = |\vec{V}|\,cos(\alpha - \theta), \text{ and}$$

$$a_2' = |\vec{V}|\,sin(\alpha - \theta).$$

If we expand these two trigonometric functions using basic identities, we arrive at:

$$\begin{aligned} \frac{a_1'}{|\vec{V}|} &= \cos\alpha\cos\theta + \sin\alpha\sin\theta \\ &= \frac{a_1}{|\vec{V}|}\cos\theta + \frac{a_2}{|\vec{V}|}\sin\theta \end{aligned},$$

and

$$\begin{aligned} \frac{a_2'}{|\vec{V}|} &= \sin\alpha\cos\theta - \cos\alpha\sin\theta \\ &= \frac{a_2}{|\vec{V}|}\cos\theta - \frac{a_1}{|\vec{V}|}\sin\theta \end{aligned}$$

, where the $\begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix} = \begin{bmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

A common mistake in applying these rotational transformations, is to lose sight of whether we are rotating the vector itself with respect to a fixed co-ordinate system or the other way around. In the case above the counter-clockwise rotation of the co-ordinate system produces a negative sign in front of the lower-left component.

Be careful to distinguish rotation of a vector about a fixed co-ordinate system and rotation of the co-ordinate system, about the fixed vector. Also be careful to note whether you are dealing with either a right-handed or a left-handed system because the signs of several of the components will change. Also note that a counter-clockwise sense is determined with respect to the basis vector while looking in the direction of its tail toward its head.

Proof of Invariance of Rotation Transformation Using Indicial Notation

Q. Show that the dot product is invariant under a rotation transformation. Show that the following is equal:

$$\vec{V}' \cdot \vec{W}' = \vec{V} \cdot \vec{W} \text{ , or}$$

$$\boldsymbol{V}_i \boldsymbol{W}_i' = \boldsymbol{V}_i \boldsymbol{W}_i \quad \text{(in indicial notation)}$$

Where the vectors in the new coordinate system, after the rotation are denoted using primes, and the vectors in the old coordinate system are plain.

Start by noting from a previous section (->) that a rotation transformation is written as follows:

$$\boldsymbol{V}' = \boldsymbol{T}\boldsymbol{V}$$

and

$$\boldsymbol{V}_i' = \boldsymbol{T}_{ij} \boldsymbol{V}_j \quad \text{(in indicial notation), where}$$

$$\boldsymbol{T}_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \boldsymbol{i,j} = 1,2$$

Also note the correspondence between actual trigonometric values and the general indices, for later reference to this proof:

$$\boldsymbol{T}_{ij} = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} \end{pmatrix}$$

Then, if we use the same rotation tensor ($\boldsymbol{T}_{ij}$) ,

$$\boldsymbol{W}_i' = \boldsymbol{T}_{ij}\boldsymbol{W}_j$$

Now,

$$\begin{aligned}
\boldsymbol{V}_i'\boldsymbol{W}_i' &= \boldsymbol{T}_{ij}\boldsymbol{V}_j\boldsymbol{T}_{ij}\boldsymbol{W}_j \\
&= \left(\boldsymbol{T}_{ij}\right)^2 \boldsymbol{V}_j\boldsymbol{W}_j \\
&= \sum_{j=1}^{j=3}\left(\boldsymbol{T}_{ij}\right)^2 \boldsymbol{V}_j\boldsymbol{W}_j \\
&= \left(\boldsymbol{T}_{i1}\right)^2 \boldsymbol{V}_1\boldsymbol{W}_1 + \left(\boldsymbol{T}_{i2}\right)^2 \boldsymbol{V}_2\boldsymbol{W}_2 + \left(\boldsymbol{T}_{i3}\right)^2 \boldsymbol{V}_3\boldsymbol{W}_3
\end{aligned}$$

The values for ***i*** also can be 1,2 or 3 as can the ***j*** values. On the LHS we must add over the values of ***i*** as per the summation convention, thus limiting the number of terms on the RHS to only 9 terms.

$$\begin{aligned}
\boldsymbol{V}_i'\boldsymbol{W}_i' &= \boldsymbol{V}_1'\boldsymbol{W}_1' + \boldsymbol{V}_2'\boldsymbol{W}_2' + \boldsymbol{V}_3'\boldsymbol{W}_3' \\
&= \left(\boldsymbol{T}_{11}\right)^2 \boldsymbol{V}_1\boldsymbol{W}_1 + \left(\boldsymbol{T}_{12}\right)^2 \boldsymbol{V}_2\boldsymbol{W}_2 + \left(\cancel{\boldsymbol{T}_{13}}\right)^2 \boldsymbol{V}_3\boldsymbol{W}_3 \\
&\quad + \left(\boldsymbol{T}_{21}\right)^2 \boldsymbol{V}_1\boldsymbol{W}_1 + \left(\boldsymbol{T}_{22}\right)^2 \boldsymbol{V}_2\boldsymbol{W}_2 + \left(\cancel{\boldsymbol{T}_{23}}\right)^2 \boldsymbol{V}_3\boldsymbol{W}_3 \\
&\quad + \left(\cancel{\boldsymbol{T}_{31}}\right)^2 \boldsymbol{V}_1\boldsymbol{W}_1 + \left(\cancel{\boldsymbol{T}_{32}}\right)^2 \boldsymbol{V}_2\boldsymbol{W}_2 + \left(\cancel{\boldsymbol{T}_{33}}\right)^2 \boldsymbol{V}_3\boldsymbol{W}_3
\end{aligned}$$

Remember that, since the rotation is only two-dimensional, some of the terms do not exist. We now have that,

$$\begin{pmatrix} \boldsymbol{T}_{11}{}^2 & \boldsymbol{T}_{12}{}^2 \\ \boldsymbol{T}_{21}{}^2 & \boldsymbol{T}_{22}{}^2 \end{pmatrix} = \begin{pmatrix} cos\,\theta^2 & sin\,\theta^2 \\ sin\,\theta^2 & cos\,\theta^2 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{V}_i'\boldsymbol{W}_i' &= cos^2\,\theta \boldsymbol{V}_1\boldsymbol{W}_1 + sin^2\,\theta \boldsymbol{V}_2\boldsymbol{W}_2 + \\
&\quad + sin^2\,\theta \boldsymbol{V}_1\boldsymbol{W}_1 + cos^2\,\theta \boldsymbol{V}_2\boldsymbol{W}_2 + \\
&= \boldsymbol{V}_1\boldsymbol{W}_1 + \boldsymbol{V}_2\boldsymbol{W}_2 \\
&= \boldsymbol{V}_i\boldsymbol{W}_i
\end{aligned}$$

(QED)

Here is a 3-D example of a three-dimensional rotation, for rotation of the X and Y axes about the Z axis (fixed) in a counter-clockwise direction (for a left-handed system):

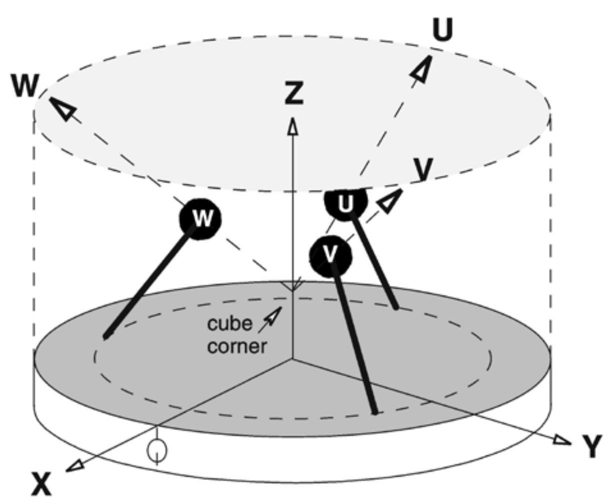
$$\begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can generalize the rotation into three dimensions. Any arbitrary rotation of an old Cartesian coordinate system into a new one can be accomplished with 3 angles, known as the Euler angles (Box 2.4 in Ilke and Amundsen, p.28)

In all cases, the length of the vector remains unchanged, although it has new co-ordinate values

An important use in earthquake seismological for the rotation of a coordinate system, involves restoring the inclined measurements in an STS-2 seismometer, This seismometer has 3 orthonormal sensors, arranged in a corner-cube geometry whose edges lie at 35.3 degrees from the horizontal (Wielandt, 2009) (from Figure 5.13), or 54.7 degrees from the vertical.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 & 1 & 1 \\ 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$



(Fig 5.13, Wielandt, 2009)

2.3 Vector multiplication

Vectorial multiplication is of two types, and both can produce either a <u>scalar</u> or a <u>vector</u> on output. There are different names you will see for each such as:

A scalar product, dot product, $\cdot$, or inner product, when applied to two vectors produces a scalar.
A scalar product between the 'del' operator (gradient or *grad)* and a scalar field, i.e., $\vec{\nabla}$(scalar) produces a vector field.
A scalar product between 'del' operator (divergence or *div*), i.e., $\vec{\nabla}$(vector) produces a vector field
Vector product, cross-product, $\times$, ; rot,*curl:* $\vec{\nabla}\times$(vector) all produce vectors

2.3.1Scalar Vector Product

There are different symbols used to denote each of these operations. Given two identical vectors $\vec{u}_1$ and $\vec{u}_2$ their scalar product is indicated as $\vec{V}\cdot\vec{W}$. We can also write this with different notation, such as $\langle \boldsymbol{V},\boldsymbol{W}\rangle$, or $\boldsymbol{V}\cdot\boldsymbol{W}$

We will try to represent vectors in matrix notation from here on, i.e.:

$$\boldsymbol{V}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \text{ and } \boldsymbol{W}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},$$

so that the result of the scalar product is represented as the sum of the products of the coefficients of the basis vectors, i.e.

$$\boldsymbol{V}\cdot\boldsymbol{W}=\left(a_1b_1+a_2b_2+a_3b_3\right).$$

Notice that the end result is not a vector but rather, a single value. We can also show that this resultant value can be evaluated if we know the length of each vector $|\boldsymbol{V}|,|\boldsymbol{W}|$ and the angle between those two vectors, $\boldsymbol{\theta}$, so that the value of the dot product can be calculated in a new way:

$$\boldsymbol{V}\cdot\boldsymbol{W}=|\boldsymbol{V}||\boldsymbol{W}|\cos\theta$$

One application of the scalar product is to determine the intensity of a sunlight on a surface and can be used to create a sense of sloping surface in a topography map. For example, given a vector that represents the normal to the land surface, and a vector than represents the sun's rays, when the two vectors are parallel to each other, theta is 0, and the following has a maximum value (=1):

$$\frac{\boldsymbol{V}\cdot\boldsymbol{W}}{|\boldsymbol{V}||\boldsymbol{W}|}=\cos\theta$$

On the other hand, when a scarp is perfectly at right angles to the sun overhead, the same expression has a value of 0. Values between 1 and 0 can then be scaled to represent full illumination of total shadow.

The scalar product between two vectors can be represented more easily in terms of indicial notation:

$$\boldsymbol{V}_i\boldsymbol{W}_i=\sum_{i=1}^{i=3}\boldsymbol{V}_i\boldsymbol{W}_i \quad \textit{(Indicial Notation)}$$

->Kronecker Delta

A scalar product between the 'del operator', or gradient operator, (represented by the Greek capital 'nabla': ∇) and a scalar field is also known as the gradient or 'grad' and produces a vector.

Example 1: A digital elevation model topographic data set consists of a scalar field—elevation values (z) at each point (x,y). The direction of the maximum slope at each point is the gradient and the value of the slope is the length of the vector.

$$\nabla z(x,y)=\frac{\partial z(x,y)}{\partial x}\hat{\boldsymbol{x}}+\frac{\partial z(x,y)}{\partial y}\hat{\boldsymbol{y}}$$

For example:

$$\boldsymbol{z}=sin(x)+cos(y)$$
$$\nabla\boldsymbol{z}=cos(x)\hat{\boldsymbol{x}}-sin(y)\hat{\boldsymbol{y}},$$

Q. What is the scalar product of any two different basis vectors

A. 0

Why?: By definition, basis vectors in a Cartesian system have a unit length and are orthogonal, so that the angle between them is 90° (cos 90°= 0) and the product is 0.

Q. What is the scalar product of any two identical basis vectors?

A. 1

Derive the answer:

In **Matlab** the cross and dot products of two vectors are calculated as shown:

```
a = [1 2 3];

b = [4 5 6];

c = cross(a,b)

c = -3     6    -3

d = dot(a,b)

d = 32

d = a*b'

d = 32
```

2.3.2 Vector Cross-Product

The other way of multiplying vectors, called a vector product, is written as follows:

$$\boldsymbol{V} \times \boldsymbol{W} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

And is expanded by determinants (see above)

In indicial notation, a cross product between two vectors is written as:

$$\boldsymbol{V}_i = \boldsymbol{\varepsilon}_{ijk} \boldsymbol{V}_j \boldsymbol{W}_k$$ *->Permutation Tensor*

In Matlab™ the cross products of two vectors are calculated as shown:

a = [1 2 3];

b = [4 5 6];

c = cross(a,b)

c =-3   6   -3

2.4 Indicial Notation/Einstein summation notation/Einstein Notation

Indicial notation by Ricci-Curbastro and Levi-Civita (1897) and its adaptation in the development of relativity (Einstein, 1916)helps express complicated tensors in a more convenient way. When using this convention, a repeated index implies addition. For example, the dot product of two vectors:

$$\boldsymbol{V}_i \boldsymbol{W}_i = \sum_{i=1}^{i=3} \boldsymbol{V}_i \boldsymbol{W}_i$$

$$\begin{aligned} a_i w_i &= \sum_{i=1}^{i=3} a_i w_i \\ &= a_1 w_1 + a_2 w_2 + a_3 w_3 \end{aligned}$$

In this type of notation, a comma signifies derivation with respect to the following index value. The summation convention always continues to apply. So, for the following case:

$$\boldsymbol{u}_{i,j} = \frac{\partial \boldsymbol{u}_i}{\partial x_j}, \quad i,j = 1,2,3$$

$$= \begin{pmatrix} \frac{\partial \boldsymbol{u}_1}{\partial x_1} & \frac{\partial \boldsymbol{u}_1}{\partial x_2} & \frac{\partial \boldsymbol{u}_1}{\partial x_3} \\ \frac{\partial \boldsymbol{u}_2}{\partial x_1} & \frac{\partial \boldsymbol{u}_2}{\partial x_2} & \frac{\partial \boldsymbol{u}_3}{\partial x_3} \\ \frac{\partial \boldsymbol{u}_3}{\partial x_1} & \frac{\partial \boldsymbol{u}_3}{\partial x_2} & \frac{\partial \boldsymbol{u}_3}{\partial x_3} \end{pmatrix}$$

For example, in indicial notation, the product of a tensor matrix and a vector, would be written as follows:

$$\begin{aligned} \boldsymbol{u}_{ji}\boldsymbol{w}_i &= \boldsymbol{w}_i\boldsymbol{u}_{ji} \\ &= \sum_{i=1}^{3} \boldsymbol{w}_i\boldsymbol{u}_{ji} \\ &= \boldsymbol{w}_1\boldsymbol{u}_{j1} + \boldsymbol{w}_2\boldsymbol{u}_{j2} + \boldsymbol{w}_3\boldsymbol{u}_{j3} \end{aligned}$$

By convention, here we summed on the repeated indices. No we can estimate all the possible combinations of *j=1,2,3,* that is:

$$\begin{aligned} \boldsymbol{u}_{1j}\boldsymbol{w}_j &= \boldsymbol{u}_{11}\boldsymbol{w}_1 + \boldsymbol{u}_{12}\boldsymbol{w}_2 + \boldsymbol{u}_{13}\boldsymbol{w}_3, \quad i = 1 \\ \boldsymbol{u}_{2j}\boldsymbol{w}_j &= \boldsymbol{u}_{21}\boldsymbol{w}_1 + \boldsymbol{u}_{22}\boldsymbol{w}_2 + \boldsymbol{u}_{23}\boldsymbol{w}_3, \quad i = 2 \\ \boldsymbol{u}_{3j}\boldsymbol{w}_j &= \boldsymbol{u}_{31}\boldsymbol{w}_1 + \boldsymbol{u}_{22}\boldsymbol{w}_2 + \boldsymbol{u}_{33}\boldsymbol{w}_3, \quad i = 3 \end{aligned}$$

We can show how this notation relates to other matrix notations by the following comparison:

$$\boldsymbol{u}_{ij}\boldsymbol{w}_j = \begin{pmatrix} \boldsymbol{u}_{11} & \boldsymbol{u}_{12} & \boldsymbol{u}_{13} \\ \boldsymbol{u}_{21} & \boldsymbol{u}_{22} & \boldsymbol{u}_{23} \\ \boldsymbol{u}_{31} & \boldsymbol{u}_{31} & \boldsymbol{u}_{33} \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \end{pmatrix}$$ or, equivalently,

$$\boldsymbol{u}_{ij}\boldsymbol{w}_j = \begin{pmatrix} \boldsymbol{u}_{11} & \boldsymbol{u}_{12} & \boldsymbol{u}_{13} \\ \boldsymbol{u}_{21} & \boldsymbol{u}_{22} & \boldsymbol{u}_{23} \\ \boldsymbol{u}_{31} & \boldsymbol{u}_{31} & \boldsymbol{u}_{33} \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_{11} \\ \boldsymbol{w}_{21} \\ \boldsymbol{w}_{31} \end{pmatrix},$$ so that when there is only one column we express the multiplication as follows:

$$\boldsymbol{u}_{ji}\boldsymbol{w}_{i1} = \boldsymbol{w}_{i1}\boldsymbol{u}_{ji},$$

i.e., ***w*** has only one column.

For the more general case, the number of rows of the first matrix may be different to the number of rows in the second matrix. (But, as always, the number of columns in the first matrix on the left must equal the number of rows in the second matrix on the right).

2.5 Determinants

A matrix is a display of numbers (Boas, p. 87). Only a square matrix can have a determinant. Vector cross product is estimated with the use of determinants. The determinant of a matrix can be evaluated by expanding it into minors with the appropriate accompanying sign (cofactor):

For example,

$$det(Y) = det\begin{pmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

$$= \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \hat{x}_1 \begin{vmatrix} a_2 & a_3 \\ w_2 & w_3 \end{vmatrix} - \hat{x}_2 \begin{vmatrix} a_1 & a_3 \\ w_1 & w_3 \end{vmatrix} + \hat{x}_3 \begin{vmatrix} a_1 & a_2 \\ w_1 & w_2 \end{vmatrix}$$

On the R.H.S., each coefficient next to basis is known as a 'minor' determinant of the principal determinant. The <u>sign</u> of each of the terms on the R.H.S. are called <u>co-factors</u>, which are determined using the following mnemonic:

Cofactor of row i and column j *(C_{ij})* is the <u>sign before the minor</u>

$$C_{ij} = -1^{i+j} M_{ij}, \text{ where}$$

$$M_{ij} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The value of the determinant is the same whether you carry out the procedure above along one row or carry out the analogous procedure along one column. The procedure is called "Laplace's development of a determinant". There are many useful facts about determinants and matrices that can be used to simplify the arrays of numbers and eventually determine the solution of sets of simultaneous equations, but for now these useful facts are beyond the scope of this class.

In the case of an nth order determinant we can use Laplace's procedure until we arrive at 2^{nd} order determinants (as above).

Depending on your prior training, determinants can be calculated using different algorithms, that is by rows (above), columns or diagonals (Sarrus' Rule or Sarrus' Scheme)

Try to calculate this determinant by hand, and in **Matlab** or **Mathematica or Excel**:

$$\begin{vmatrix} 0 & 6 & 3 & 5 \\ 2 & 8 & 9 & 4 \\ 1 & 5 & 11 & 4 \\ 2 & 0 & 0 & 1 \end{vmatrix} = ?$$

2.6 Kronecker Delta in Indicial Notation

Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For example, in the case of the dot product (scalar product) which we saw previously, (*) we noted that the indicial notation for:

$$\boldsymbol{V}_i \boldsymbol{W}_i = \sum_{i=1}^{i=3} \boldsymbol{V}_i \boldsymbol{W}_i$$

Use of the Kronecker delta allows us to rewrite this also as:

$$\boldsymbol{V}_i\boldsymbol{W}_i = \boldsymbol{V}_i\boldsymbol{W}_j\delta_{ij}$$

We no longer have repeated indices, so that we must find all possible combinations of i and j, but also consider the qualification by the Kronecker delta that can null the value of product. This may be seen more readily if we expand the above expression:

$$\boldsymbol{V}_i\boldsymbol{W}_j\delta_{ij} =$$

$$\begin{Bmatrix} \boldsymbol{V}_1\boldsymbol{W}_1 + \boldsymbol{V}_1\boldsymbol{W}_2 + \boldsymbol{V}_1\boldsymbol{W}_3 + \ldots \\ \boldsymbol{V}_2\boldsymbol{W}_1 + \boldsymbol{V}_2\boldsymbol{W}_2 + \boldsymbol{V}_2\boldsymbol{W}_3 + \ldots \\ \boldsymbol{V}_3\boldsymbol{W}_1 + \boldsymbol{V}_3\boldsymbol{W}_2 + \boldsymbol{V}_3\boldsymbol{W}_3 \end{Bmatrix}\delta_{ij} \qquad i,j = 1,2,3$$

$$\boldsymbol{V}_i\boldsymbol{W}_j\delta_{ij} =$$

$$\begin{Bmatrix} \boldsymbol{V}_1\boldsymbol{W}_1 + \cancel{\boldsymbol{V}_1\boldsymbol{W}_2} + \cancel{\boldsymbol{V}_1\boldsymbol{W}_3} + \ldots \\ \cancel{\boldsymbol{V}_2\boldsymbol{W}_1} + \boldsymbol{V}_2\boldsymbol{W}_2 + \cancel{\boldsymbol{V}_2\boldsymbol{W}_3} + \ldots \\ \cancel{\boldsymbol{V}_3\boldsymbol{W}_1} + \cancel{\boldsymbol{V}_3\boldsymbol{W}_2} + \boldsymbol{V}_3\boldsymbol{W}_3 \end{Bmatrix}$$

Remember, that the magnitude of this vector that results from this dot-product multiplication of two vectors is also:

$$\boldsymbol{V}_i\boldsymbol{W}_j\delta_{ij} = \left|\vec{V}\cdot\vec{W}\right|$$

$$\left|\vec{V}\cdot\vec{W}\right| = \left|\vec{V}\right|\left|\vec{W}\right| cos\,\theta$$

2.7 Permutation Tensor/Levi-Civita Permutation Tensor and Indicial Notation

We are introducing here an advanced form of indicial notation, known as the permutation tensor (or alternating tensor) which is a skew-symmetric tensor, that is that the off-diagonal terms are equal and opposite, e.g.,

$$\boldsymbol{T}_{ij} = -\boldsymbol{T}_{ji}$$

The Levi-Civita, permutation or alternating tensor has the following definition for n-dimensions (Wrede, 1972):

$$\varepsilon_{1......r..n} = \begin{cases} +1 \; if \; sub-indices \; are \; even \\ -1 \, if \; sub-indices \; are \; odd \\ \quad 0, \quad otherwise \end{cases}$$

We can see more readily how the alternating tensor works if we work from low-dimension determinants to higher-dimension determinants.

Let us start by calculating the determinant a where a has two rows ($j=1,2$) and two columns ($k=1,2$), that is, let's work out the following:

$$\det a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\det a = a_{11}a_{22} - a_{21}a_{12}$$

Using, indicial notation the determinant of a can also be expressed as:

$$\det a = \varepsilon_{jk} a_{1j} a_{2k}$$

If we use the summation convention over repeated indices we can sum systematically over the complete range of values for each index; first over j's

$\det a = \varepsilon_{1k} a_{11} a_{2k} + \varepsilon_{2k} a_{12} a_{2k}$, and then over the k's:

$$\det a = \varepsilon_{11} a_{11} a_{21} + \varepsilon_{12} a_{11} a_{22} + \varepsilon_{21} a_{12} a_{21} + \varepsilon_{22} a_{11} a_{22}$$

Now, we apply the convention for the value of the alternating tensor to create the coefficients of all the a-based terms:

$$\det a = \varepsilon_{11}(=0) a_{11} a_{21} + \varepsilon_{12}(=1) a_{11} a_{22} + \varepsilon_{21}(=-1) a_{12} a_{21} + \varepsilon_{22}(=0) a_{11} a_{22}$$

Finally, there are only two non-zero terms left:

$$\det a = a_{11} a_{22} - a_{12} a_{21}$$

We have revealed the compact and accurate nature of the alternating tensor.

If we increase the dimensions of the determinant by one we will have that:

$$\det a = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

If we do this one more time, for a 4th-order determinant:

$$\det a = \varepsilon_{hijk} a_{1h} a_{2i} a_{3j} a_{4k}$$

So that for much higher dimensions, the *n*th order determinant can by symbolized as follows:

$$\det a = \begin{vmatrix} a_{11} & . & . & . & a_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

$$\det a = \varepsilon_{p_1 p_2 \cdots p_r \cdots p_n} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_r q_r} \cdots a_{p_n q_n},$$

Where $p_{r=1,2,3,4\ldots l}$, $q_{r=1,2,3,4\ldots l}$

In the case of second-order tensors, we will need to use the permutation tensor in the following situations:

$\varepsilon_{ijk} = 1$ for *ijk* or *jki* or *kij*

e.g., $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

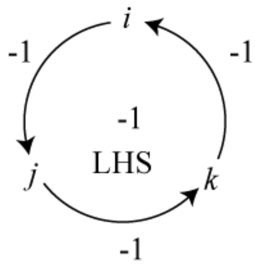
Also, $\varepsilon_{ijk} = -1$ for *jik* or *ikj* or *kji*

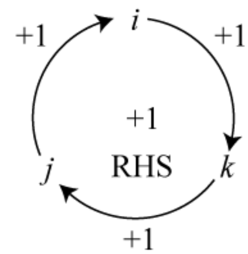
e.g., $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$

$\varepsilon_{ijk} = 0$ when any index is repeated

e.g., $\varepsilon_{112} = \varepsilon_{121} = \varepsilon_{211} = \varepsilon_{221} \quad etc. \quad = 0$

Use the following diagrams as mnemonics for determining the sign of the permutation tensor.





When mixing different generic indices remember these equivalences:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = 1$$

$$-\varepsilon_{ijk} = \varepsilon_{ikj} = \varepsilon_{kji} = -1$$

$$\varepsilon_{kii} = \varepsilon_{ijj} = \varepsilon_{kki} = 0$$

Let's show the use of the permutation tensor for abbreviating a cross-product between two vectors (->).

$$\boldsymbol{X}_i = \varepsilon_{ijk}\boldsymbol{V}_j\boldsymbol{W}_k$$

$$\boldsymbol{V}\times\boldsymbol{W} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\left(\boldsymbol{V}\times\boldsymbol{W}\right)_i = \varepsilon_{ijk}\boldsymbol{V}_j\boldsymbol{W}_k$$

Example uses of Permutation Tensor

The following identities for the permutation tensor can be very useful:

1. $\varepsilon_{ijk}\varepsilon_{ijk} = 6$
2. $\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}$
3. $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

Proof for identity 1:

$$\varepsilon_{ijk} = \begin{cases} +1 \ \mathit{if}\ (i,j,k)\ \mathit{is}\ (1,2,3),(2,3,1)\ \mathit{or}\ (3,1,2) \\ -1\ \mathit{if}\ (i,j,k)\ \mathit{is}\ (2,1,3),(3,2,1)\ \mathit{or}\ (1,3,2) \\ \ 0 \quad \mathit{otherwise} : i = j, j = k\ \mathit{or}\ i = k \end{cases}$$

So in all the 27 possibilities, only the 6 shown above is none zero

So

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{ijk} &= 1\times 1+1\times 1+1\times 1 \\ &\quad +(-1)\times(-1)+(-1)\times(-1)+(-1)\times(-1) \\ &= 6 \end{aligned}$$

Let's show the usefulness of the permutation tensor by way of an example that demonstrates the following vectorial identity:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$$

Notice that (1) the final result will be a vector and that (2) ultimately, we want to obtain the elements of a vector (e.g., k=1,2,3)

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\left(\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})\right)_k$$

$$\varepsilon_{ijk}\boldsymbol{A}_j(\boldsymbol{B}\times\boldsymbol{C})_k=\varepsilon_{ijk}\boldsymbol{A}_j\varepsilon_{klm}\boldsymbol{B}_l\boldsymbol{C}_m$$

$i,j,k,,l$ and m are called dummy variables which can by equal to 1, 2 or 3 at any time i.e.,

i,j,k =1,2,3 , and

k,l,m =1,2,3

We continue to follow the convention that we must sum over repeated indices.

At this point we can incorporate one of the identities from above, so that we now have

$$
\begin{aligned}
\varepsilon_{ijk}\boldsymbol{A}_j\varepsilon_{klm}\boldsymbol{B}_l\boldsymbol{C}_m &= \left(\delta_{il}\delta_{jm}-\delta_{jl}\delta_{im}\right)\boldsymbol{A}_j\boldsymbol{B}_l\boldsymbol{C}_m \\
&= \underbrace{\delta_{il}\delta_{jm}\boldsymbol{A}_j\boldsymbol{B}_l\boldsymbol{C}_m}_{(I)} - \underbrace{\delta_{jl}\delta_{im}\boldsymbol{A}_j\boldsymbol{B}_l\boldsymbol{C}_m}_{(II)}
\end{aligned}
$$

We can now expand each of these terms on the right to see if there are any terms that can be excluded from the derivation.

There are too many terms to try to do this in a brute force way, so let's first ask **four** big-picture questions which cover all four possibilities.

Q1. In so doing, we should think in which cases the combination of indices will provide either *I=0 OR II=0* ?

i.e., when will *I=0?.....* OR when will *II=0?.....*

For the case of the first term on the right,

I=0 ? when $i \neq l \ \mid\ j \neq m$

II=0 ? when $j \neq l \ \mid\ i \neq m$

Q.2 Let us ask a second question before we get confused by the terms.

In which cases, will both terms on the right be equal, i.e.,

I=II ? when

$i=l \quad \& \quad j=m \quad \& \quad j=l \quad \& \quad i=m$,

i.e., $i=l=j=m$, when all the indices are simultaneously equal.

For this case ***I-II=0 (!!)***, whether they are individually equal to 0 or not.

Q.3 Finally, for which case will *I=0* AND *II=0 ?*

I=II=0? when $i \neq l \ \mid\ j \neq m$ AND $j \neq l \ \mid\ i \neq m$

Q. 4 Now, the only cases that will be left, will be those where

$I \neq 0\, AND\, II \neq 0$, (both are non-zero)

$I \neq 0\,?$ when $i = l \quad \& \, j = m$

$II \neq 0\,?$ when $j = l \quad \& \, i = m$

$I \neq 0 \; OR \; II \neq 0$ (i.e., only one of ther two terms is non-zero)

Now thanks to this overview, we know that the only cases which will contribute are those where either terms on the right are not equal to 0 or to each other.

So, when is $I \neq 0$ and $i = l \quad \& \, j = m$ we obtain the following expression:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{lmk} \boldsymbol{A}_m \boldsymbol{\varepsilon}_{klm} \boldsymbol{B}_l \boldsymbol{C}_m &= \left(\delta_{ll}\delta_{mm} - \delta_{ml}\delta_{lm}\right) \boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m \\
\left(\boldsymbol{\varepsilon}_{lmk}{}^2 \boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m\right) &= \delta_{ll}\delta_{mm} \boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m - \cancel{\delta_{ml}\delta_{lm} \boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m} \\
\left(\boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m\right) &= \boldsymbol{A}_m \boldsymbol{B}_l \boldsymbol{C}_m \\
&= \boldsymbol{A}_m \boldsymbol{C}_m \boldsymbol{B}_l \\
&= \left(\boldsymbol{A} \cdot C\right) \boldsymbol{B}_l \\
&\qquad (I)
\end{aligned}
$$

(Recall from earlier in these notes that the scalar product in indicial notation

is

$$
\boldsymbol{V}_i \boldsymbol{W}_j \delta_{ij} = \left|\vec{V} \cdot \vec{W}\right|
$$

So, when $II \neq 0, \quad j = l \quad \& \, i = m$ we obtain the following expression:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{mlk} \boldsymbol{A}_l \boldsymbol{\varepsilon}_{klm} \boldsymbol{B}_l \boldsymbol{C}_m &= \left(\delta_{ml}\delta_{lm} - \delta_{ll}\delta_{mm}\right) \boldsymbol{A}_l \boldsymbol{B}_l \boldsymbol{C}_m \\
&= \cancel{\delta_{ml}\delta_{lm} \boldsymbol{A}_l \boldsymbol{B}_l \boldsymbol{C}_m} - \delta_{ll}\delta_{mm} \boldsymbol{A}_l \boldsymbol{B}_l \boldsymbol{C}_m \\
&= -\delta_{ll}\delta_{mm} \boldsymbol{A}_l \boldsymbol{B}_l \boldsymbol{C}_m \\
&= \left(-\delta_{ll}\delta_{mm} \boldsymbol{A}_l \boldsymbol{B}\right) \boldsymbol{C}_m \\
&= -\boldsymbol{A}_l \boldsymbol{B}_l \boldsymbol{C}_m \\
&= -(\boldsymbol{A} \cdot \boldsymbol{B}) \boldsymbol{C}_m \\
&\qquad (II)
\end{aligned}
$$

(QED)

2.6 General Matrix Multiplication: expanding the indicial notation

In general matrix multiplication, we multiply over the columns and add over the rows. Different to dot and cross products of vectors we can multiply matrices of variable dimensions. The only restriction is that the number of columns (*j*) in the first matrix is equal to the number of rows (*i*) in the second matrix during the multiplication.

Using indicial notation, for two different matrices, ***A*** and ***B***,

$$C_{ij} = A_{ik}B_{kj} \,,$$

Where, for matrix *A*, *i* is the number or rows and *k* the number of columns. For matrix *B*, *i* is the number or rows and *k* the number of columns. For matrix C, *i* is the number or rows and *k* the number of columns.

In order to be able to multiply matrices, the <u>number of columns in the first matrix</u> must equal the <u>number of rows in the second matrix</u> (Hence *k* is repeated). Note that *i* and *j* can be different and that in the following example we are assuming a special case where they are equal to 3.

According to indicial notation, since the *k* index is repeated then there must be a summation between the multiplications of columns of the first matrix and rows of the second. So, in a more expanded form we obtain:

$$C_{ij} = \sum_{k=1}^{3} A_{ik}B_{kj}$$

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j}$$

If we now consider every possible combination of *i* and *j*, where *i=1,2,3* and *j=1,2,3* we have 9 permutations (tensor rank=3 (=dimension), order =2)

$$\begin{aligned}
C_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\
C_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\
C_{13} &= A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \\
C_{21} &= A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\
C_{22} &= A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\
C_{23} &= A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\
C_{31} &= A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} \\
C_{32} &= A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \\
C_{33} &= A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33}
\end{aligned}$$

These permutations can be placed into a matrix, consisting of 3 rows and 3 columns

$$C_{ij} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

In Matlab™:

```
>> A= [1 0 1;0 0 0]

A = 1     0     1

      0     0     0

>> B= [2 0 3;1 3 3; 1 2 2]

B =

    2     0     3

    1     3     3

    1     2     2

>> A*B

ans = ?
```

Q. Do this example above by hand

* See Powerpoint for examples of matrix Multiplication in matlab

References

Maxwell, S. C., Rutledge, J., Jones, R., and Fehler, M., 2010, Petroleum reservoir characterization using downhole microseismic monitoring: Geophysics, v. 75, no. 5, p. 75A129-175A137.

Wielandt, E., 2009, Seismic sensors and their calibration, *in* Bormann, P., ed., New Manual of Seismological Observatory Practice, Volume 1: Potsdam, GeoForschungsZentrum.

Exercises in vector and tensor calculus

1. In the case the indices are 1,2,3 show that the following is true:

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

2. Also show that the indicial notation for a 4th-order or 4x4 determinant, where h,i,j,and k is true:

$$\det a = \varepsilon_{hijk} a_{1h} a_{2i} a_{3j} a_{4k}$$

3. Given a ¼ spherical surface with the accompanying matlab program, calculate the gradient at each point on the sphere and plot out the gradient as a vector plot viewed from above.

For your convenience I have added example Matlab code that generates and plots ¼ of a spherical surface:

```
% program to plot sphere
% Z2 = -x*x - y*y + radius^2
% Declare the dimensions
% to reserve space
radius= 50;
X1 = ones (100,100);
Y1 = ones (100,100);
Z1 = ones (100,100);

x = 1:1:100;
y = (1:1:100)';

X = X1 .* repmat(x,100,1)
Y = Y1 .* repmat(y,1,100)

% repeat vectors vertically 100 times
X2 =  X .* X
Y2 =  Y .* Y
Z2 =   -X2 - Y2 + radius*radius
Z  =   sqrt(Z2) .* Z1
```

```
% zero out areas outside the sphere where solutions are not
% real
zero_out = find(Z2<0)
Z(zero_out) = 0

surf(X,Y,Z)




%Z
% radius=1
%
% z = sqrt(z2)
```