Petroleum Seismology, spring 2010
Homework #1

Purpose: **Review Vector Algebra**

Q. 1 For three given vectors, **a,b,c,** show that the following identities hold true

$$\begin{aligned}
&\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}\\
&\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}\\
&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\\
&\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})\\
&\nabla\times(\nabla\times\boldsymbol{a})=\nabla(\nabla\cdot\boldsymbol{a})-\nabla^2\boldsymbol{a}
\end{aligned} \tag{1}$$

where $\nabla^2=\nabla\cdot\nabla$

Let us check for the 3-d case:
Assume:

$$\vec{a}=a_1\overrightarrow{x_1}+a_2\overrightarrow{x_2}+a_3\overrightarrow{x_3}$$

$$\vec{b}=b_1\overrightarrow{x_1}+b_2\overrightarrow{x_2}+b_3\overrightarrow{x_3}$$

$$\vec{c}=c_1\overrightarrow{x_1}+c_2\overrightarrow{x_2}+c_3\overrightarrow{x_3}$$

1) $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$

Proof:

$$\vec{a}\bullet\vec{b}=a_1b_1+a_2b_2+a_3b_3$$

$$\vec{b}\bullet\vec{a}=b_1a_1+b_2a_2+b_3a_3$$

$$\Rightarrow \vec{a}\bullet\vec{b}=\vec{b}\bullet\vec{a}$$

2) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

Proof:

$$\vec{a} \times \vec{b}$$

$$= \begin{vmatrix} \overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \overrightarrow{x_1}(a_2 b_3 - a_3 b_2) - \overrightarrow{x_2}(a_1 b_3 - a_3 b_1) + \overrightarrow{x_3}(a_1 b_2 - a_2 b_1)$$

$$\vec{b} \times \vec{a}$$

$$= \begin{vmatrix} \overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \overrightarrow{x_1}(b_2 a_3 - b_3 a_2) - \overrightarrow{x_2}(b_1 a_3 - b_3 a_1) + \overrightarrow{x_3}(b_1 a_2 - b_2 a_1)$$

$$= -\overrightarrow{x_1}(a_2 b_3 - a_3 b_2) + \overrightarrow{x_2}(a_1 b_3 - a_3 b_1) - \overrightarrow{x_3}(a_1 b_2 - a_2 b_1)$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

3) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$

Proof:

$$\vec{b} \times \vec{c}$$

$$= \begin{vmatrix} \overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \overrightarrow{x_1}(b_2c_3 - b_3c_2) - \overrightarrow{x_2}(b_1c_3 - b_3c_1) + \overrightarrow{x_3}(b_1c_2 - b_2c_1)$$

$$\vec{a} \times (\vec{b} \times \vec{c})$$

$$= \begin{vmatrix} \overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= \overrightarrow{x_1}(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))$$
$$- \overrightarrow{x_2}(a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2))$$
$$+ \overrightarrow{x_3}(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))$$

$$(\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\overrightarrow{x_1} + b_2\overrightarrow{x_2} + b_3\overrightarrow{x_3})$$
$$- (a_1b_1 + a_2b_2 + a_3b_3)(c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3})$$

$$
\begin{aligned}
&= \overrightarrow{x_1}(b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3)) \\
&\quad + \overrightarrow{x_2}(b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)) \\
&\quad + \overrightarrow{x_3}(b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3)) \\
&= \overrightarrow{x_1}(b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)) \\
&\quad + \overrightarrow{x_2}(b_2(a_1c_1 + a_3c_3) - c_2(a_1b_1 + a_3b_3)) \\
&\quad + \overrightarrow{x_3}(b_3(a_1c_1 + a_2c_2) - c_3(a_1b_1 + a_2b_2)) \\
&= \overrightarrow{x_1}(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)) \\
&\quad - \overrightarrow{x_2}(a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)) \\
&\quad + \overrightarrow{x_3}(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))
\end{aligned}
$$

$$
\Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{c}) \times \vec{b} - (\vec{a} \bullet \vec{b}) \times \vec{c}
$$

4) $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$

Proof:

$$\vec{b}\times\vec{c}$$

$$=\begin{vmatrix}\overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\overrightarrow{x_1}(b_2c_3-b_3c_2)-\overrightarrow{x_2}(b_1c_3-b_3c_1)+\overrightarrow{x_3}(b_1c_2-b_2c_1)$$

$$\vec{a}\bullet(\vec{b}\times\vec{c})$$

$$=(a_1\overrightarrow{x_1}+a_2\overrightarrow{x_2}+a_3\overrightarrow{x_3})\bullet(\overrightarrow{x_1}(b_2c_3-b_3c_2)-\overrightarrow{x_2}(b_1c_3-b_3c_1)+\overrightarrow{x_3}(b_1c_2-b_2c_1))$$

$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

$$\vec{c}\times\vec{a}$$

$$=\begin{vmatrix}\overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3}\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}$$

$$=\overrightarrow{x_1}(c_2a_3-c_3a_2)-\overrightarrow{x_2}(c_1a_3-c_3a_1)+\overrightarrow{x_3}(c_1a_2-c_2a_1)$$

$$\vec{b}\bullet(\vec{c}\times\vec{a})$$

$$=b_1(c_2a_3-c_3a_2)-b_2(c_1a_3-c_3a_1)+b_3(c_1a_2-c_2a_1)$$

$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

$$\Rightarrow\vec{a}\bullet(\vec{b}\times\vec{c})=\vec{b}\bullet(\vec{c}\times\vec{a})$$

The same calculation will apply to the third one, I will not show it here

5) $\nabla\times(\nabla\times\boldsymbol{a})=\nabla(\nabla\cdot\boldsymbol{a})-\nabla^2\boldsymbol{a}$

Proof:

$$\nabla\times\vec{a}$$

$$=\begin{vmatrix}\overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ a_1 & a_2 & a_3\end{vmatrix}$$

$$=\overrightarrow{x_1}\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)-\overrightarrow{x_2}\left(\frac{\partial a_3}{\partial x_1}-\frac{\partial a_1}{\partial x_3}\right)+\overrightarrow{x_3}\left(\frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\right)$$

$$\nabla\times(\nabla\times\vec{a})$$

$$=\begin{vmatrix}\overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ \frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3} & \frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1} & \frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\end{vmatrix}$$

$$=\overrightarrow{x_1}\begin{vmatrix}\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ \frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1} & \frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\end{vmatrix}-\overrightarrow{x_2}\begin{vmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3}\\ \frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3} & \frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\end{vmatrix}+\overrightarrow{x_3}\begin{vmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\\ \frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3} & \frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1}\end{vmatrix}$$

$$=\overrightarrow{x_1}\left(\frac{\partial^2 a_2}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_1\partial x_3}-\frac{\partial^2 a_1}{\partial {x_2}^2}-\frac{\partial^2 a_1}{\partial {x_3}^2}\right)+\overrightarrow{x_2}\left(\frac{\partial^2 a_1}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_2\partial x_3}-\frac{\partial^2 a_2}{\partial {x_1}^2}-\frac{\partial^2 a_2}{\partial {x_3}^2}\right)$$

$$+\overrightarrow{x_3}\left(\frac{\partial^2 a_1}{\partial x_1\partial x_3}+\frac{\partial^2 a_2}{\partial x_2\partial x_3}-\frac{\partial^2 a_3}{\partial {x_1}^2}-\frac{\partial^2 a_3}{\partial {x_2}^2}\right)$$

$$\nabla\left(\nabla\cdot\vec{a}\right)=\nabla\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\right)$$

$$=\vec{x}_1\left(\frac{\partial^2 a_2}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_1\partial x_3}+\frac{\partial^2 a_1}{\partial x_1{}^2}\right)+\vec{x}_2\left(\frac{\partial^2 a_1}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_2\partial x_3}+\frac{\partial^2 a_2}{\partial x_2{}^2}\right)$$

$$+\vec{x}_3\left(\frac{\partial^2 a_1}{\partial x_1\partial x_3}+\frac{\partial^2 a_2}{\partial x_2\partial x_3}+\frac{\partial^2 a_3}{\partial x_3{}^2}\right)$$

$$\nabla^2\vec{a}=\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_3{}^2}\right)\left(a_1\vec{x}_1+a_2\vec{x}_2+a_3\vec{x}_3\right)$$

$$=\vec{x}_1\left(\frac{\partial^2 a_1}{\partial x_1{}^2}+\frac{\partial^2 a_1}{\partial x_2{}^2}+\frac{\partial^2 a_1}{\partial x_3{}^2}\right)+\vec{x}_2\left(\frac{\partial^2 a_2}{\partial x_1{}^2}+\frac{\partial^2 a_2}{\partial x_2{}^2}+\frac{\partial^2 a_2}{\partial x_3{}^2}\right)$$

$$+\vec{x}_3\left(\frac{\partial^2 a_3}{\partial x_1{}^2}+\frac{\partial^2 a_3}{\partial x_2{}^2}+\frac{\partial^2 a_3}{\partial x_3{}^2}\right)$$

$$\Rightarrow\nabla\left(\nabla\cdot\vec{a}\right)-\nabla^2\vec{a}$$

$$=\vec{x}_1\left(\frac{\partial^2 a_2}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_1\partial x_3}-\frac{\partial^2 a_1}{\partial x_2{}^2}-\frac{\partial^2 a_1}{\partial x_3{}^2}\right)+\vec{x}_2\left(\frac{\partial^2 a_1}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_2\partial x_3}-\frac{\partial^2 a_2}{\partial x_1{}^2}-\frac{\partial^2 a_2}{\partial x_3{}^2}\right)$$

$$+\vec{x}_3\left(\frac{\partial^2 a_1}{\partial x_1\partial x_3}+\frac{\partial^2 a_2}{\partial x_2\partial x_3}-\frac{\partial^2 a_3}{\partial x_1{}^2}-\frac{\partial^2 a_3}{\partial x_2{}^2}\right)$$

$$=\nabla\times\left(\nabla\times\vec{a}\right)$$

Q 2 Express, in matrix form, the rotation of vector b by 90^0 clockwise. Do include a diagram to illustrate the rotation.

Solution:
For a general case, let us assume we rotate the vector by θ clockwise, here is the diagram

We have

$$a_1 = |v|\cos\alpha$$
$$a_2 = |v|\sin\alpha$$
$$a_1' = |v|\cos(\alpha+\theta)$$
$$= |v|(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$= a_1\cos\theta - a_2\sin\theta$$
$$a_2' = |v|\sin(\alpha+\theta)$$
$$= |v|(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$
$$= a_1\sin\theta + a_2\cos\theta$$
$$\Rightarrow \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$For\, \theta = 90^\circ, \sin\theta = 1, \cos\theta = 0$$
$$a_1' = a_1\cos\theta - a_2\sin\theta = -a_2$$
$$a_2' = a_1\sin\theta + a_2\cos\theta = a_1$$

Please see the right for the diagram.