

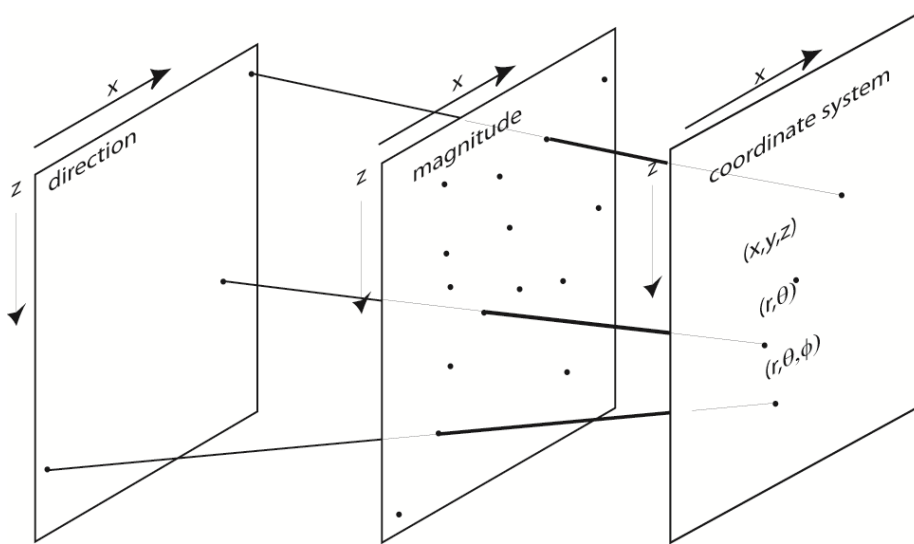
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## TENSORS

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Tensors are abstract objects describable by **arrays of functions**. Each function of such an array is also called a **component**. Components are functions of the selected co-ordinate system (Hadsell, 1995). As such, the component can change in aspect with the change of reference system. However, the value of the property they serve to represent does not.

Let's try to visualize the components of a tensor and their relation to real physical values they represent:



A tensor is called an  **$n$ th order tensor** when it comprises an array of  $r^n$  components, where  $r$  is the number of the dimension (2D, 3D, etc.) in use and  $n$  is called the order. We will be mainly seeing second or third-order tensors in three or four dimensions, so that our arrays can have from  $3^2$  to  $4^3$  components. Arrays with nine components can be written in the form of a  $3 \times 3$  matrix. Arrays with 81 components are more difficult to visualize because they are described by a three-dimensional lattice array  $3 \times 3 \times 3$ . In the case of more than two dimensions indicial notation becomes a very convenient way to deal with tensors.

### STRESS TENSOR

A stress tensor is one of order 2 and in 3 dimensional space has 9 components.

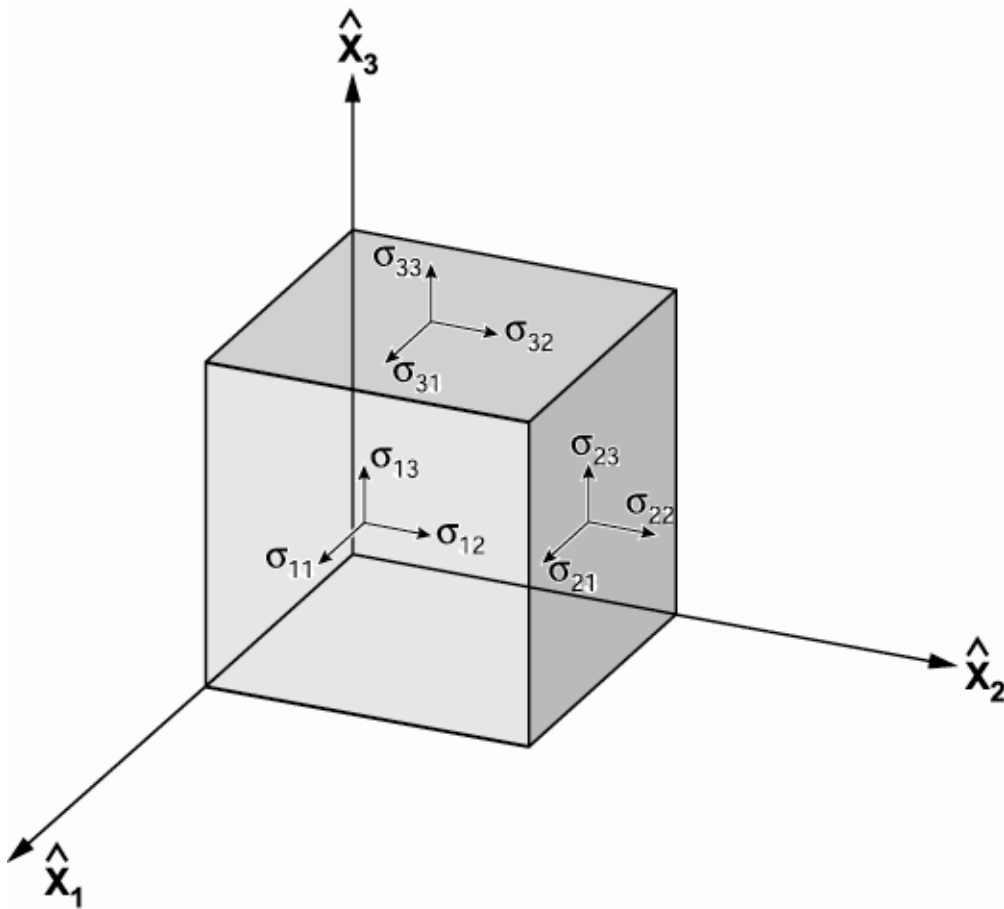
In order to study and define the stress at a point we can study stress in two ways. First, on three orthogonal planes passing through the point, or we can study stress on the six sides of an infinitesimally small cube as the cube tends to that point.

We view this cube as being acted upon by body forces on all the particles of mass it contains. Body forces act at a distance and could include a gravitational, electrical or

magnetic field that is acting on the body. It is convenient for us to disregard these body forces for now assuming that the body is not experiencing any linear acceleration that can contribute to a gradient in the stress vector field.

We also see this small cube as non-rotating with no net torque acting upon it.

In a Eulerian view of the world we fix our point of observation, say at a geophone and measure how quickly the ground moves with respect to the fixed point. Another approach to viewing the world would be to fix our reference frame upon the particle in motion and describe the wavefield from the new vantage point (Lagrangian view). However, in many cases and in our entire approach we will use a Eulerian view of the world.



Stress on a surface can be treated as a “stress vector” and is also called a **traction vector** ( $\vec{T}$ ) Traction is defined as:

$$\vec{T} = \lim_{\Delta A \rightarrow 0} \frac{\Delta \vec{F}}{\Delta A}$$

Using indicial notation,  $\Delta \vec{F}$  can be written as  $F_i = \Delta F_i x_i$

In the limit, traction equals force per unit area at a point on one face. (We take as convention that the surface stress acts on the outer surfaces of the cube upon the inner surface.)

Each face of the cube can experience traction or stress. We use the following convention to denote stress:

$$\sigma_{ij}$$

where  $i$  is the index of the axis to which the face is normal and  $j$  is the index of the direction in which the component of the traction vector is applied. When the component of the traction vector is applied in the direction of the basis vector, our convention is to use a positive value and negative when the component is applied opposite to the sense of the basis vector. Sometimes

$\sigma_{ij} = \sigma_{(i)i}$  and each component is called normal traction component. e.g.,

$$\sigma_1, \sigma_2, \sigma_3$$

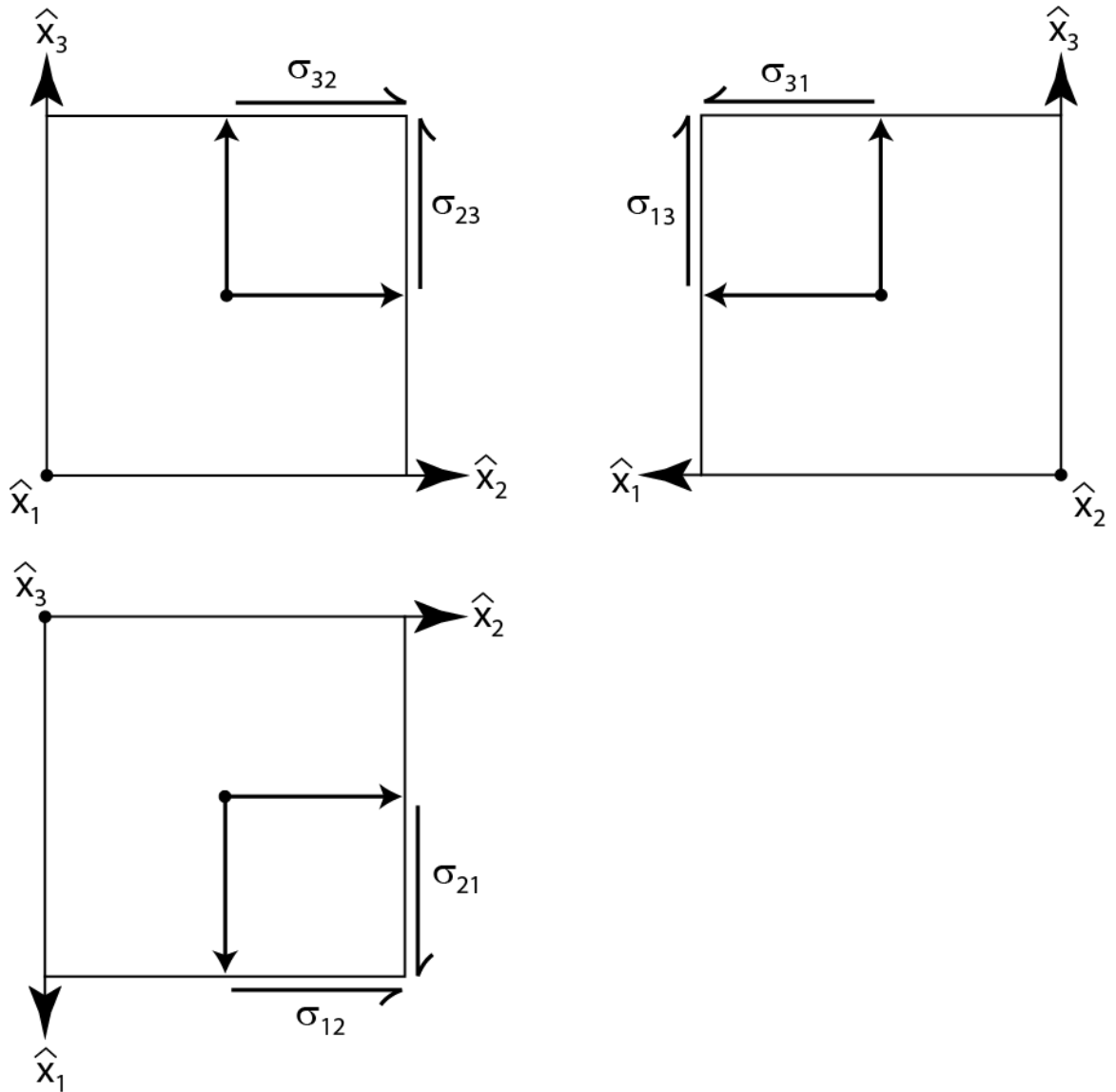
Often, when  $i \neq j$ ,  $\sigma_{ij} = \tau_{ij}$  and  $\tau_{ij}$  being used to denote the shear traction component. Ikelle and Amundsen (2005) use  $\tau_{ij}$  for both normal and shear traction components.

If the cube does not rotate then the shear tractions must cancel each other out, so that for each case, the net torque is zero, where

$$\text{Torque} = \mathbf{r} \times \mathbf{F}$$

$$|\text{Torque}| = |\mathbf{r}| |\mathbf{F}| \sin \theta$$

We obtain force from stress by multiplying the stress over the area it acts. Force is the product of traction times the surface area over which the traction is applied.



If the cube in question does not experience a net rotation, because the material is elastic, then the net torque should be 0.

Of the 18 traction components, there are three pairs that must cancel each other's effects if we assume that there is no net rotation during the application of stress. For the three cases we get:

$$\hat{x}_1 (\sigma_{21} dx_1 dx_3) \times r \hat{x}_2 = -\hat{x}_2 (\sigma_{12} dx_2 dx_3) \times r \hat{x}_1$$

$$\hat{x}_3 (\sigma_{23} dx_3 dx_1) \times r \hat{x}_2 = -\hat{x}_2 (\sigma_{32} dx_2 dx_1) \times r \hat{x}_3$$

$$\hat{x}_3 (\sigma_{13} dx_3 dx_2) \times r \hat{x}_1 = -\hat{x}_1 (\sigma_{31} dx_1 dx_2) \times r \hat{x}_3$$

In indicial notation these three cases can be written as:

$$\mathcal{E}_{ijk} (\sigma_{21} dx_1 dx_3, 0, 0)_j (0, r, 0)_k = -\mathcal{E}_{ijk} (0, \sigma_{12} dx_2 dx_3, 0)_j (r, 0, 0)_k$$

$$\mathcal{E}_{ijk} (0, 0, \sigma_{23} dx_3 dx_1)_j (0, r, 0)_k = -\mathcal{E}_{ijk} (0, \sigma_{32} dx_2 dx_1, 0)_j (0, 0, r)_k$$

$$\mathcal{E}_{ijk} (0, 0, \sigma_{13} dx_3 dx_2)_j (r, 0, 0)_k = -\mathcal{E}_{ijk} (\sigma_{31} dx_1 dx_2, 0, 0)_j (0, 0, r)_k$$

For the equation to hold true then  $\sigma_{ji} = \sigma_{ij}$ , which is the description of a **symmetric tensor**. In other words, in order for the net torque to be zero the tensor must be symmetric.

A tensor,  $t_{ij}$  is symmetric iff

$$\mathcal{E}_{ijk} t_{jk} = 0.$$

The symmetry means that the off-diagonal terms are equal

$$\begin{aligned} \mathcal{E}_{ijk} t_{jk} &= \sum_{j=1}^{j=3} \mathcal{E}_{ijk} t_{jk} \\ &= \sum_{k=1}^{k=3} (\mathcal{E}_{i1k} t_{1k} + \mathcal{E}_{i2k} t_{2k} + \mathcal{E}_{i3k} t_{3k}) \\ &= \mathcal{E}_{i11} t_{11} + \mathcal{E}_{i12} t_{12} + \mathcal{E}_{i13} t_{13} + \\ &\quad \mathcal{E}_{i21} t_{21} + \mathcal{E}_{i22} t_{22} + \mathcal{E}_{i23} t_{23} + \\ &\quad \mathcal{E}_{i31} t_{31} + \mathcal{E}_{i32} t_{32} + \mathcal{E}_{i33} t_{33} \end{aligned}$$

For example, if  $i=1$ :

$$\begin{aligned}
\mathcal{E}_{1jk}t_{jk} &= \sum_{j=1}^{j=3} \mathcal{E}_{1jk}t_{jk} \\
&= \sum_{k=1}^{k=3} (\mathcal{E}_{11k}t_{1k} + \mathcal{E}_{12k}t_{2k} + \mathcal{E}_{13k}t_{3k}) \\
&= \mathcal{E}_{111}t_{11} + \mathcal{E}_{112}t_{12} + \mathcal{E}_{113}t_{13} + \\
&\quad \mathcal{E}_{121}t_{21} + \mathcal{E}_{122}t_{22} + \mathcal{E}_{123}t_{23} + \\
&\quad \mathcal{E}_{131}t_{31} + \mathcal{E}_{132}t_{32} + \mathcal{E}_{133}t_{33} \\
&= 0 + 0 + 0 + \\
&\quad 0 + 0 + t_{23} + \\
&\quad 0 - t_{32} + 0
\end{aligned}$$

Remember that

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & \textcircled{t_{23}} \\ t_{31} & \textcircled{t_{32}} & t_{33} \end{pmatrix}$$

An essential working assumption for this theorem is that the body must be in mechanical equilibrium, that is, the body is not experiencing any change in its linear or angular momentum. The invariance of the value of the stress field reminds us of the general invariance properties we examined earlier with vectors, and is hinted at by the symmetry of the general stress tensor that we demonstrated.

#### CAUCHY'S THEOREM OR STRESS PRINCIPLE (BEN- MENAHEM)

Cauchy's theorem states that given a plane of interest, we can define a traction vector (a first-order tensor) at a point ( $\vec{T}(\vec{n})$ ) on this plane in terms of any orthonormal reference system. That means that although the stress field tensor is a second-order tensor we can deal with stress as a lower-order tensor, simplifying our mathematical complexity. In other words, according to Cauchy's theorem and using indicial notation we can have

$$\mathbf{T}_i = \sigma_{ji} \mathbf{n}_j$$

In other words, stress at a point can be manipulated equally well as if it were only a vector without loss of accuracy in our description of the real world.

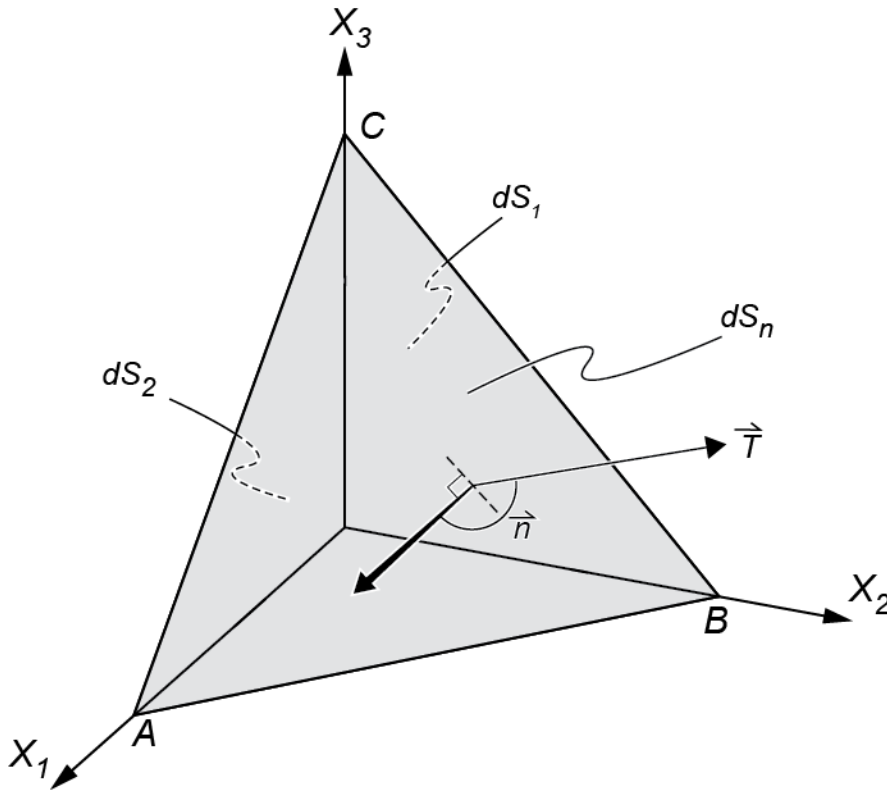
Especially note the presence of the components  $\mathbf{n}_i$ . The Cauchy theorem requires that the traction vector be a function of a plane, which is described by the direction cosine components ( $n_i$ ) of the **unit** vector ( $\vec{n}$ ) normal to the plane. In summary:

$$n_i = \cos \theta_i$$

$$n_i n_i = 1 = (\cos \theta_i)(\cos \theta_i)$$

The angle  $\theta$  is the angle between each component of the unit vector  $\mathbf{n}$  and the corresponding coordinate axis.

We can demonstrate Cauchy's theorem by balancing the forces on the sloping face of a tetrahedron against the forces on the other three sides. This tetrahedron in the limit goes to a point as its dimensions become infinitesimally small. If the equilibrium condition (no net torque) is met we will discover that Cauchy's theorem holds true.



It will be very helpful to note before we begin that

$$dS_i = dS_n (\cos \theta_i),$$

e.g., 
$$dS_1 = dS_n [\cos(\mathbf{n}, \hat{x}_1)]$$

where  $\cos \theta_i$  is the direction cosine with regard to the  $\hat{x}_i$  basis vector.

We begin by balancing forces:

$$\begin{aligned}\int \vec{T}(\vec{n})dS &= \vec{T}(\vec{n})dS_n + \vec{T}(-\hat{x}_1)dS_1 + \vec{T}(-\hat{x}_2)dS_2 + \vec{T}(-\hat{x}_3)dS_3 \\ &= \vec{T}(\vec{n})dS_n - \vec{T}(\hat{x}_1)dS_1 - \vec{T}(\hat{x}_2)dS_2 - \vec{T}(\hat{x}_3)dS_3\end{aligned}$$

Note that  $\vec{T}(\hat{x}_1)$  is a three-component stress vector across the  $x_1$  plane, which has a normal  $\hat{x}_1$ , and is multiplied by  $dS_1$  a scalar (units of area), so that each term on the right hand side is also a vector scaled by the value of the surface across which the stress vector is acting.

Given that the tetrahedron is in equilibrium, then  $\int \vec{T}(\vec{n})dS = 0$ , so that the first term on the right-hand-side must equal the sum of the remaining three, i.e.

$$\vec{T}(\vec{n})dS_n = \vec{T}(\hat{x}_1)dS_1 + \vec{T}(\hat{x}_2)dS_2 + \vec{T}(\hat{x}_3)dS_3.$$

In other words, if we know the stress on three planes of the tetrahedron (RHS) we can tell the stress on a general plane (LHS)!

By substituting the scalar value of the surface area of each side of the tetrahedron ( $dS_i$ ) in terms of the surface area of the main face  $dS_n$ , we have:

$$\vec{T}(\vec{n})dS_n = \vec{T}(\hat{x}_1)dS_n \left( \frac{\vec{n} \cdot \hat{x}_1}{|\vec{n}|} \right) + \vec{T}(\hat{x}_2)dS_n \left( \frac{\vec{n} \cdot \hat{x}_2}{|\vec{n}|} \right) + \vec{T}(\hat{x}_3)dS_n \left( \frac{\vec{n} \cdot \hat{x}_3}{|\vec{n}|} \right)$$

Rearranging terms, we have:

$$\vec{T}(\vec{n})dS_n = \frac{\vec{n}}{|\vec{n}|} \left( \vec{T}(\hat{x}_1) \cdot \hat{x}_1 + \vec{T}(\hat{x}_2) \cdot \hat{x}_2 + \vec{T}(\hat{x}_3) \cdot \hat{x}_3 \right) dS_n$$

Dividing both sides by  $dS_n$ , we have:

$$\mathbf{T}_i(\vec{n}) = \frac{n_i}{|\vec{n}|} T_j(\hat{x}_i) \hat{x}_j$$

Each component of the general stress tensor on the left-hand side of the equation comprises components of stress of all three other faces.

As an example, the first component of the stress vector in the  $\hat{x}_1$  direction on a face perpendicular to the  $\hat{x}_2$  direction is  $T_1(\hat{x}_2)\hat{x}_1$ , or  $\sigma_{21}$ . The first component of stress vector in the  $\hat{x}_2$  direction on a face perpendicular to the  $\hat{x}_2$  direction is  $T_1(\hat{x}_2)\hat{x}_2$ , or  $\sigma_{22}$ . The first component of the stress vector in the  $\hat{x}_3$  direction on a face perpendicular to the  $\hat{x}_2$  direction is  $T_1(\hat{x}_2)\hat{x}_3$ , or  $\sigma_{23}$ .

The complete stress vector on this face has three components:

$$\begin{aligned} \mathbf{T}_2(\vec{n}) &= \vec{T}(\hat{x}_2) \\ &= \mathbf{n}_2(T_1(\hat{x}_2)\hat{x}_1 + T_2(\hat{x}_2)\hat{x}_2 + T_3(\hat{x}_2)\hat{x}_3), \text{ or} \\ \vec{T}(\hat{x}_2) &= \mathbf{n}_2(\sigma_{21}\hat{x}_1 + \sigma_{22}\hat{x}_2 + \sigma_{23}\hat{x}_3) \end{aligned}$$

as expressed in the geological convention. From here we can further generalize the example and examine the contribution from each face to the  $\hat{x}_1$  component of the stress vector on the general face, which we have shown is the result of summing the contributions on the rest of the tetrahedron:

$$\mathbf{T}_1(\hat{x}_1) = (\mathbf{n}_1\sigma_{11} + \mathbf{n}_2\sigma_{21} + \mathbf{n}_3\sigma_{31})\hat{x}_1$$

In indicial notation, all the cases can be summarized as:

$$\mathbf{T}_i = \mathbf{n}_j \sigma_{ji}$$

-> Acoustic Wave Equation

where  $\sigma_{ji}$  is the general stress tensor and  $\mathbf{n}_j$  is the component of the normal to the plane.

Some examples of stress include:

(1) Hydrostatic stress:

$$\begin{pmatrix} \rho gh & 0 & 0 \\ 0 & \rho gh & 0 \\ 0 & 0 & \rho gh \end{pmatrix}$$

,where  $\rho$  is the earth's density,  $g$  gravitational acceleration and  $h$  depth in the earth's crust.

(2) uni-directional stress (say only in the  $\hat{x}_1$ ) superimposed on lithostatic stress

$$\begin{pmatrix} \rho gh + s_{11} & 0 & 0 \\ s_{21} & \rho gh & 0 \\ s_{31} & 0 & \rho gh \end{pmatrix}$$

(3) In geological circles, general stress is always representable simply as:

$$\begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

