
VECTORS AND THE SYMMETRY OF PHYSICAL LAWS

FIELDS

SYMMETRY

VECTORS

Invariance of Vectors under Linear Transformation

Counter-clockwise rotation of a Cartesian Coordinate System

Some Advanced Properties of Vectors and Scalars

Vector multiplication

Scalar Vector Product

Properties of Vector Scalar products--TBD

Determinants

Vector Product

Properties of Vector Cross-products--TBD

General Matrix Multiplication

Einstein/Indicial Notation

Kronecker Delta in Indicial Notation

Permutation Tensor and Indicial Notation

Field: Is a physical quantifiable property that can be defined over some n-dimensional space

Symmetry: (paraphrase: Hermann Weyl: a thing is symmetrical when after undergoing mathematical operations it looks the same as when we started, e.g. rotation of a plain undecorated vase by 180 degrees)

FIELDS

We talk of the electromagnetic field as a description of electrical and magnetic properties within a given space. We talk of the gravitational field to describe gravity values and the direction of this force throughout a given body in space and time. When we say the elastic wave field we describe the elastic behavior of earth materials to the passing of a wave by how fast they oscillate (Hz) and how much they move (amplitude in m).

Each different field of physical properties has a different complexity that can be described with increasingly complex, and more general, mathematics. If a property varies as a function ONLY of its position in space, i.e.

$$= f(x_1, x_2, x_3),$$

then the property and field is known as scalar.

Examples of different types of fields:

SCALAR FIELDS: Density field, temperature field, salinity field, [Poisson's ratio](#) field, shear modulus field, Young's modulus field

VECTOR FIELDS: [Velocity field](#), Heat flow field, diffusivity of sediment field, gravity field, displacement,

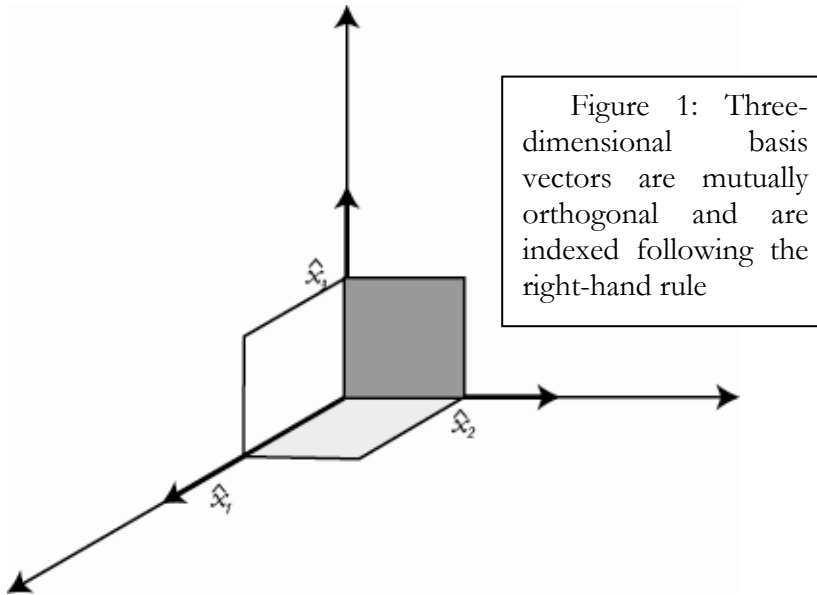
TENSOR FIELDS: Stress, strain, thermal conductivity

SYMMETRY (FEYNMAN LECTURES, CH. 11)

Although it may seem obvious, certain physical laws do not change even if we modify our co-ordinate systems. That is, if we have our origin in one place and observe a wave field but then we choose to describe the wave field from a different origin or a different fixed frame of reference the observation will be the same. This wave field is symmetric.

VECTORS

Vectors are quantities that have a directional property as well as a value in space. With a vector we know HOW MUCH it is worth and whether this quantity acts in a certain direction. A vector is described using three numbers.



Basis or unitary vectors in a cartesian co-ordinate system are mutually orthogonal (orthonormal) to each other, are of unit length and obey the right-hand rule. We can describe these vectors in several ways:

$$\vec{V} = a_1\hat{x}_1 + a_2\hat{x}_2 + a_3\hat{x}_3$$

or

$$\vec{V} = a_i\hat{x}_i$$

(If indices are repeated by convention we sum over them)

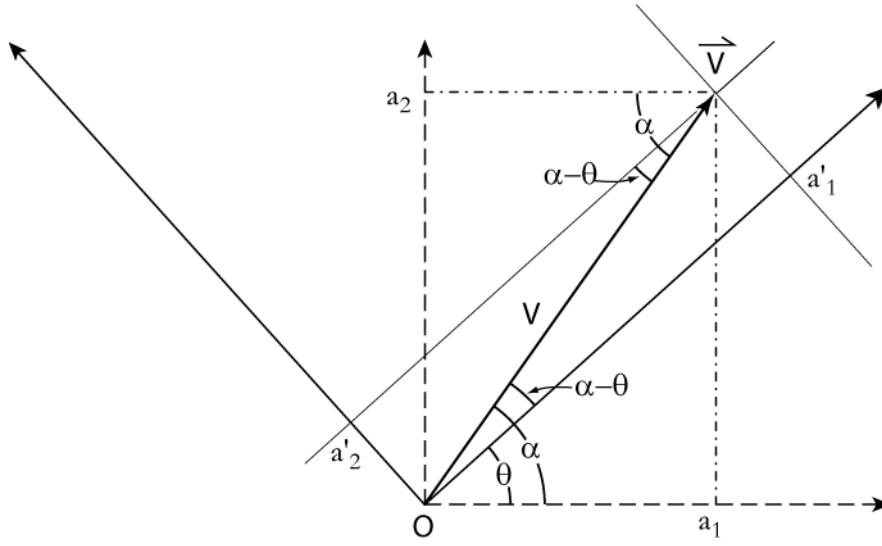
Invariance of Vectors under Linear Transformation

A vector property is also a property that is symmetric. That is, it does not matter whether these three numbers are different as measured with respect to different origins or frames of reference. These different numbers will still describe the same physical behavior, of say, the wave field.

Example 1: For example, if a vector is describing the velocity of a wave field at the earth's surface in a direction that is not perpendicular to the earth's surface we may choose to more conveniently rotate the co-ordinate reference frame in line with the particle motion. This happens in cases when studying anisotropy where we sometimes try out different rotations of the reference system until we maximize the wave energy coming from a

particular direction. Example 2: In hydraulic fracturing, one manner for determining the direction of the originating microseism is to rotate the coordinate system until the direction in which particle motion of a particular wave mode is maximized, also known as the 'hodogram method' (In (Maxwell et al., 2010).

Counter-clockwise rotation of a Cartesian Coordinate System (Left-handed system)



$$\begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ or, more briefly expressed as}$$

$$\mathbf{V}' = \mathbf{T}\mathbf{V}, \quad \text{--> Proof of Invariance using Indicical Notation}$$

where \mathbf{V}' is the vector after the transformation expressed in components in terms of the rotated co-ordinate basis vectors and \mathbf{V} is the vector before the transformation expressed in terms of components of the unaffected basis vector system.

Before the co-ordinate rotation the co-ordinates for \mathbf{V} are:

$$a_1 = |\vec{V}| \cos \alpha, \text{ and } a_2 = |\vec{V}| \sin \alpha$$

After the co-ordinate rotation the co-ordinates for \mathbf{V}' are:

$a'_1 = |\vec{V}| \cos(\alpha - \theta)$, and $a'_2 = |\vec{V}| \sin(\alpha - \theta)$. Expanding these two trigonometric functions using [basic identities](#) we arrive at:

$$\begin{aligned}\frac{a'_1}{|\vec{V}|} &= \cos \alpha \cos \theta + \sin \alpha \sin \theta \\ &= \frac{a_1}{|\vec{V}|} \cos \theta + \frac{a_2}{|\vec{V}|} \sin \theta\end{aligned},$$

and

$$\begin{aligned}\frac{a'_2}{|\vec{V}|} &= \sin \alpha \cos \theta - \cos \alpha \sin \theta \\ &= \frac{a_2}{|\vec{V}|} \cos \theta - \frac{a_1}{|\vec{V}|} \sin \theta\end{aligned}$$

$$\text{When the } \begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

A common mistake in applying these rotational transformations, is to lose sight of whether we are rotating the vector itself with respect to a fixed co-ordinate system or the other way around. In the above case the counter-clockwise rotation of the co-ordinate system produces a negative sign in front of the lower-left component.

Be careful to distinguish rotation of a vector about a fixed co-ordinate system and rotation of the co-ordinate system, about the fixed vector. Also be careful to note whether you are dealing with either a right-handed or a left-handed system because the signs of several of the components will change. Also note that a counter-clockwise sense is determined with respect to the basis vector while looking in the direction of its tail toward its head

Here is a 3-D example of a three-dimensional rotation, for rotation of the X and Y axes about the Z axis (fixed) in a counter-clockwise direction (for a left-handed system):

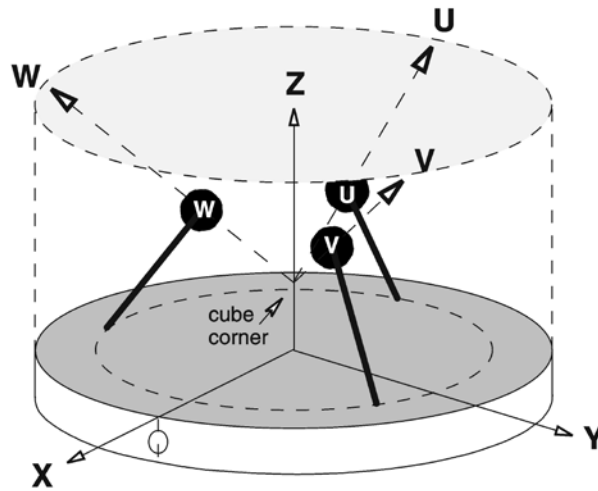
$$\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can generalize the rotation into three dimensions. Any arbitrary rotation of an old Cartesian coordinate system into a new one can be accomplished with 3 angles, known as the Euler angles (Box 2.4 in Ilke and Amundsen, p.28)

In all cases, the length of the vector remains unchanged, although it has new co-ordinate values

An important use in earthquake seismological for the rotation of a coordinate system, involves restoring the inclined measurements in an STS-2 seismometer. This seismometer has 3 orthonormal sensors, arranged in a corner-cube geometry whose edges lie at 35.3 degrees from the horizontal (Wielandt, 2009) (from Figure 5.13), or 54.7 degrees from the vertical.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 & 1 & 1 \\ 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$



(Fig 5.13, Wielandt, 2009)

Some Advanced Properties of Vectors and Scalars

Vector multiplication

Vectorial multiplication is of two types and can either produce a scalar or a vector on output. There are different names you will see for each such as:

Scalar product, dot product, \cdot , inner product, *divergence*: $\nabla \cdot$

Vector product, cross-product, \times , *curl*: $\nabla \times$,

Scalar Vector Product

There are different symbols used to denote each of these operations. Given two identical vectors \vec{u}_1 and \vec{u}_2 their scalar product is indicated as $\vec{V} \cdot \vec{W}$. We can also write this with different notation, such as $\langle \mathbf{V}, \mathbf{W} \rangle$, or $\mathbf{V} \cdot \mathbf{W}$

We will try to represent vectors in matrix notation from here on, i.e.:

$$\mathbf{V} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{W} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

so that the result of the scalar product is represented as the sum of the products of the coefficients of the basis vectors, i.e.

$$\mathbf{V} \cdot \mathbf{W} = (a_1 b_1 + a_2 b_2 + a_3 b_3).$$

Notice that the end result is not a vector but rather, a single value. We can also show that this resultant value can be evaluated if we know the length of each vector $|\mathbf{V}|, |\mathbf{W}|$ and the angle between those two vectors, θ , so that the value of the dot product can be calculated in a new way:

$$\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}| |\mathbf{W}| \cos \theta$$

$$\mathbf{V}_i \mathbf{W}_i = \sum_{i=1}^{i=3} \mathbf{V}_i \mathbf{W}_i \text{ (Indicial Notation)} \quad \text{--> Kronecker Delta}$$

A scalar product between the ‘del operator’, or gradient operator, (represented by the Greek capital ‘nabla’: ∇) and a scalar field is also known as the gradient or ‘grad’.

Example 1: A digital elevation model topographic data set consists of a scalar field—elevation values (z) at each point (x, y). The direction of the maximum slope at each point is the gradient and the value of the slope is the length of the vector.

$$\nabla z(x, y) = \frac{\partial z(x, y)}{\partial x} \hat{x} + \frac{\partial z(x, y)}{\partial y} \hat{y}$$

For example:

$$z = \sin(x) + \cos(y) \\ \nabla z = \cos(x) \hat{x} - \sin(y) \hat{y}$$

Q. What is the scalar product of any two different basis vectors

A. 1

Why?: By definition, basis vectors in a Cartesian system have a unit length and are orthogonal, so that the angle between them is 90° ($\cos 90^\circ = 0$) and the product is 0.

Q. What is the scalar product of any two identical basis vectors?

A. 1

Derive the answer:

In **Matlab** the cross and dot products of two vectors are calculated as shown:

```
a = [1 2 3];
```

```
b = [4 5 6];
```

```
c = cross(a,b)
```

```
c = -3    6   -3
```

```
d = dot(a,b)
```

```
d = 32
```

```
d = a*b'
```

```
d = 32
```

Determinants

A matrix is a display of numbers (Boas, p. 87). Only a square matrix can have a determinant. Vector cross product is estimated with the use of determinants. The determinant of a matrix can be evaluated by expanding it into minors with the appropriate accompanying sign (cofactor):

For example,

$$\det(Y) = \det \begin{pmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

$$= \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \hat{x}_1 \begin{vmatrix} a_2 & a_3 \\ w_2 & w_3 \end{vmatrix} - \hat{x}_2 \begin{vmatrix} a_1 & a_3 \\ w_1 & w_3 \end{vmatrix} + \hat{x}_3 \begin{vmatrix} a_1 & a_2 \\ w_1 & w_2 \end{vmatrix}$$

On the R.H.S., each coefficient next to basis is known as a ‘minor’ determinant of the principal determinant. The sign of each of the terms on the R.H.S. are called co-factors, which are determined using the following mnemonic:

Cofactor of row i and column j (C_{ij}) is the sign before the minor

$$C_{ij} = -1^{i+j} M_{ij}, \text{ where}$$

$$M_{ij} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The value of the determinant is the same whether you carry out the procedure above along one row or carry out the analogous procedure along one column. The procedure is called “Laplace’s development of a determinant”. There are many useful facts about determinants and matrices that can be used to simplify the arrays of numbers and eventually determine the solution of sets of simultaneous equations, but for now these useful facts are beyond the scope of this class.

In the case of an n th order determinant we can use Laplace’s procedure until we arrive at 2nd order determinants (as above).

Depending on your prior training, determinants can be calculated using different algorithms, that is by rows (above), columns or diagonals (Sarrus’ Rule or Sarrus’ Scheme)

Try to calculate this determinant by hand, and in **Matlab** or **Mathematica** or **Excel**:

$$\begin{vmatrix} 0 & 6 & 3 & 5 \\ 2 & 8 & 9 & 4 \\ 1 & 5 & 11 & 4 \\ 2 & 0 & 0 & 1 \end{vmatrix} = ?$$

Vector Product/Cross-Product

The other way of multiplying vectors, called a vector product, is written as follows:

$$\mathbf{V} \times \mathbf{W} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

And is expanded by determinants (see above)

In indicial notation, a cross product between two vectors is written as:

$$V_i = \epsilon_{ijk} V_j W_k \quad \text{-> Permutation Tensor}$$

In MatlabTM the cross products of two vectors are calculated as shown:

a = [1 2 3];

b = [4 5 6];

c = cross(a,b)

c = -3 6 -3

Properties of Vector Products

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}$$

Einstein summation notation/Einstein Notation/Indicial Notation

A good set of notes on Indicial Notation comes from Michael Raulli (pdf file on Google: **Raulli Tensors**).

Indicial notation (Einstein, 1916) expresses conveniently complicated tensors. When using this convention, a repeated index implies addition. For example, the dot product of two vectors:

$$\begin{aligned}
\mathbf{v}_i \mathbf{w}_i &= \sum_{i=1}^{i=3} \mathbf{v}_i \mathbf{w}_i \\
a_i w_i &= \sum_{i=1}^{i=3} a_i w_i \\
&= a_1 w_1 + a_2 w_2 + a_3 w_3
\end{aligned}$$

In this type of notation, a comma signifies derivation with respect to the following index value. The summation convention always continues to apply. So, for the following case:

$$\begin{aligned}
\mathbf{u}_{i,j} &= \frac{\partial \mathbf{u}_i}{\partial x_j}, \quad i, j = 1, 2, 3 \\
&= \begin{pmatrix} \frac{\partial \mathbf{u}_1}{\partial x_1} & \frac{\partial \mathbf{u}_1}{\partial x_2} & \frac{\partial \mathbf{u}_1}{\partial x_3} \\ \frac{\partial \mathbf{u}_2}{\partial x_1} & \frac{\partial \mathbf{u}_2}{\partial x_2} & \frac{\partial \mathbf{u}_2}{\partial x_3} \\ \frac{\partial \mathbf{u}_3}{\partial x_1} & \frac{\partial \mathbf{u}_3}{\partial x_2} & \frac{\partial \mathbf{u}_3}{\partial x_3} \end{pmatrix}
\end{aligned}$$

A tensor matrix and vector product, for example, would be written as follows in indicial notation:

$$\begin{aligned}
\mathbf{u}_{ji} \mathbf{w}_i &= \mathbf{w}_i \mathbf{u}_{ji} \\
&= \sum_{i=1}^3 \mathbf{w}_i \mathbf{u}_{ji} \\
&= \mathbf{w}_1 \mathbf{u}_{j1} + \mathbf{w}_2 \mathbf{u}_{j2} + \mathbf{w}_3 \mathbf{u}_{j3}
\end{aligned}$$

After, by convention, we sum on the repeated indices, we can estimate all the possible combinations of $j=1,2,3$.

Now,

$$\begin{aligned}
\mathbf{u}_{1j} \mathbf{w}_j &= \mathbf{u}_{11} \mathbf{w}_1 + \mathbf{u}_{12} \mathbf{w}_2 + \mathbf{u}_{13} \mathbf{w}_3, \quad i = 1 \\
\mathbf{u}_{2j} \mathbf{w}_j &= \mathbf{u}_{21} \mathbf{w}_1 + \mathbf{u}_{22} \mathbf{w}_2 + \mathbf{u}_{23} \mathbf{w}_3, \quad i = 2 \\
\mathbf{u}_{3j} \mathbf{w}_j &= \mathbf{u}_{31} \mathbf{w}_1 + \mathbf{u}_{32} \mathbf{w}_2 + \mathbf{u}_{33} \mathbf{w}_3, \quad i = 3
\end{aligned}$$

We can relate this notation by comparing the same estimation involving matrix notation:

$$\mathbf{u}_{ij} \mathbf{w}_j = \begin{pmatrix} \mathbf{u}_{11} & \mathbf{u}_{12} & \mathbf{u}_{13} \\ \mathbf{u}_{21} & \mathbf{u}_{22} & \mathbf{u}_{23} \\ \mathbf{u}_{31} & \mathbf{u}_{31} & \mathbf{u}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{pmatrix} \text{ or, equivalently,}$$

$$\mathbf{u}_{ij} \mathbf{w}_j = \begin{pmatrix} \mathbf{u}_{11} & \mathbf{u}_{12} & \mathbf{u}_{13} \\ \mathbf{u}_{21} & \mathbf{u}_{22} & \mathbf{u}_{23} \\ \mathbf{u}_{31} & \mathbf{u}_{31} & \mathbf{u}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{w}_{11} \\ \mathbf{w}_{21} \\ \mathbf{w}_{31} \end{pmatrix}, \text{ which is the same as:}$$

$$\mathbf{u}_{ji} \mathbf{w}_{i1} = \mathbf{w}_{i1} \mathbf{u}_{ji},$$

i.e., there is only one column.

For the more general case, the number of rows of the first matrix may be different to the number of rows in the second matrix.

Proof of Invariance of Rotation Transformation Using Indicjal Notation

Q. Show that the dot product is invariant under a rotation transformation. Show that the following is equal:

$$\vec{V}' \cdot \vec{W}' = \vec{V} \cdot \vec{W}, \text{ or}$$

$$\mathbf{V}_i \mathbf{W}'_i = \mathbf{V}_i \mathbf{W}_i \quad (\text{in indicjal notation})$$

Where the vectors in the new coordinate system, after the rotation are denoted using primes, and the vectors in the old coordinate system are plain.

Start by noting from a previous section ([->](#)) that a rotation transformation is written as follows:

$$\mathbf{V}' = \mathbf{T} \mathbf{V}$$

and $\mathbf{V}'_i = \mathbf{T}_{ij} \mathbf{V}_j$ (in indicjal notation), where

$$\mathbf{T}_{ij} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad i, j = 1, 2$$

Also note the correspondence between actual trigonometric values and the general indices, for later reference to this proof:

$$\mathbf{T}_{ij} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix}$$

Then, if we use the same rotation tensor (T_{ij}),

$$W'_i = T_{ij} W_j$$

$$\begin{aligned} V_i W'_i &= T_{ij} V_j T_{ij} W_j \\ &= (T_{ij})^2 V_j W_j \end{aligned}$$

Now,

$$\begin{aligned} &= \sum_{j=1}^{j=3} (T_{ij})^2 V_j W_j \\ &= (T_{i1})^2 V_1 W_1 + (T_{i2})^2 V_2 W_2 + (T_{i3})^2 V_3 W_3 \end{aligned}$$

The values for i also can be 1,2 or 3 as can the j values. On the LHS we must add over the values of i as per the summation convention, thus limiting the number of terms on the RHS to only 9 terms.

$$\begin{aligned} V_i W'_i &= V_1 W'_1 + V_2 W'_2 + V_3 W'_3 \\ &= (T_{11})^2 V_1 W_1 + (T_{12})^2 V_2 W_2 + (T_{13})^2 V_3 W_3 \\ &\quad + (T_{21})^2 V_1 W_1 + (T_{22})^2 V_2 W_2 + (T_{23})^2 V_3 W_3 \\ &\quad + (T_{31})^2 V_1 W_1 + (T_{32})^2 V_2 W_2 + (T_{33})^2 V_3 W_3 \end{aligned}$$

Remember that, since the rotation is only two-dimensional, some of the terms do not exist. We now have that,

$$\begin{pmatrix} T_{11}^2 & T_{12}^2 \\ T_{21}^2 & T_{22}^2 \end{pmatrix} = \begin{pmatrix} \cos^2 \theta & \sin \theta^2 \\ \sin \theta^2 & \cos \theta^2 \end{pmatrix}$$

$$\begin{aligned} V_i W'_i &= \cos^2 \theta V_1 W_1 + \sin^2 \theta V_2 W_2 + \\ &\quad + \sin^2 \theta V_1 W_1 + \cos^2 \theta V_2 W_2 + \\ &= V_1 W_1 + V_2 W_2 \\ &= V_i W_i \end{aligned}$$

(QED)

Kronecker Delta in Indicical Notation

Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For example, in the case of the dot product (scalar product) which we saw previously, (*) we noted that the indicial notation for:

$$\mathbf{V} \cdot \mathbf{W} = \sum_{i=1}^{i=3} \mathbf{V}_i \mathbf{W}_i$$

Use of the Kronecker delta allows us to rewrite this also as:

$$\mathbf{V}_i \mathbf{W}_i = \mathbf{V}_i \mathbf{W}_j \delta_{ij}$$

We no longer have repeated indices, so that we must find all possible combinations of i and j , but also consider the qualification by the Kronecker delta that can null the value of product. This may be seen more readily if we expand the above expression:

$$\mathbf{V}_i \mathbf{W}_j \delta_{ij} = \begin{Bmatrix} \mathbf{V}_1 \mathbf{W}_1 + \mathbf{V}_1 \mathbf{W}_2 + \mathbf{V}_1 \mathbf{W}_3 + \dots \\ \mathbf{V}_2 \mathbf{W}_1 + \mathbf{V}_2 \mathbf{W}_2 + \mathbf{V}_2 \mathbf{W}_3 + \dots \\ \mathbf{V}_3 \mathbf{W}_1 + \mathbf{V}_3 \mathbf{W}_2 + \mathbf{V}_3 \mathbf{W}_3 \end{Bmatrix} \delta_{ij} \quad i, j = 1, 2, 3$$

$$\mathbf{V}_i \mathbf{W}_j \delta_{ij} = \begin{Bmatrix} \mathbf{V}_1 \mathbf{W}_1 + \cancel{\mathbf{V}_1 \mathbf{W}_2} + \cancel{\mathbf{V}_1 \mathbf{W}_3} + \dots \\ \cancel{\mathbf{V}_2 \mathbf{W}_1} + \mathbf{V}_2 \mathbf{W}_2 + \cancel{\mathbf{V}_2 \mathbf{W}_3} + \dots \\ \cancel{\mathbf{V}_3 \mathbf{W}_1} + \cancel{\mathbf{V}_3 \mathbf{W}_2} + \mathbf{V}_3 \mathbf{W}_3 \end{Bmatrix}$$

Remember, that the magnitude of this vector that results from this dot-product multiplication of two vectors is also:

$$\mathbf{V}_i \mathbf{W}_j \delta_{ij} = |\vec{V} \cdot \vec{W}|$$

$$|\vec{V} \cdot \vec{W}| = |\vec{V}| |\vec{W}| \cos \theta$$

Permutation Tensor/Levi-Civita Permutation Tensor and Indicial Notation

We are introducing here an advanced form of indicial notation, known as the permutation tensor (or alternating tensor) which is a skew-symmetric tensor, that is that the off-diagonal terms are equal and opposite, e.g.,

$$\mathbf{T}_{ij} = -\mathbf{T}_{ji}$$

We define the permutation tensor as

$$\mathcal{E}_{ijk} = 1 \quad \text{for } ijk \text{ or } jki \text{ or } kij$$

e.g., $\mathcal{E}_{123} = \mathcal{E}_{231} = \mathcal{E}_{312} = 1$

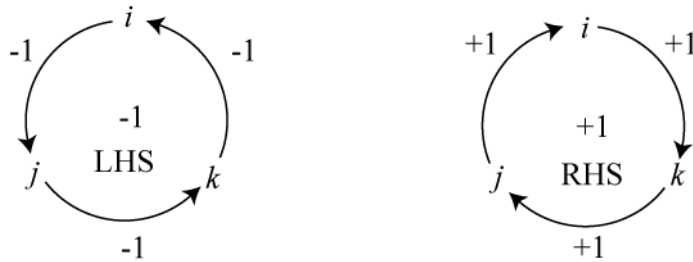
Also, $\mathcal{E}_{ijk} = -1 \quad \text{for } jik \text{ or } ikj \text{ or } kji$

e.g., $\mathcal{E}_{132} = \mathcal{E}_{213} = \mathcal{E}_{321} = -1$

$$\mathcal{E}_{ijk} = 0 \quad \text{when any index is repeated}$$

e.g., $\mathcal{E}_{112} = \mathcal{E}_{121} = \mathcal{E}_{211} = \mathcal{E}_{221} \text{ etc. } = 0$

Use the following diagrams as mnemonics for determining the sign of the permutation tensor.



When mixing different generic indices remember these equivalences:

$$\mathcal{E}_{ijk} = \mathcal{E}_{jki} = \mathcal{E}_{kij} = 1$$

$$-\mathcal{E}_{ijk} = \mathcal{E}_{ikj} = \mathcal{E}_{kji} = -1$$

$$\mathcal{E}_{kii} = \mathcal{E}_{ijj} = \mathcal{E}_{kki} = 0$$

Let's show the use of the permutation tensor for abbreviating a cross-product between two vectors ([->](#)).

$$V_i = \mathcal{E}_{ijk} V_j W_k$$

$$\mathbf{V} \times \mathbf{W} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$(\mathbf{V} \times \mathbf{W})_i = \mathcal{E}_{ijk} V_j W_k$$

The following identities for the permutation tensor can be very useful:

$$\mathcal{E}_{ijk} \mathcal{E}_{ijk} = 6$$

$$\mathcal{E}_{ijk} \mathcal{E}_{ijl} = 2\delta_{kl}$$

$$\mathcal{E}_{ijk} \mathcal{E}_{klm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

Let's show the usefulness of the permutation tensor by way of an example that demonstrates the following vectorial identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}$$

Notice that (1) the final result will be a vector and that (2) ultimately, we want to obtain the elements of the vector ($k=1,2,3$)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times (\mathbf{B} \times \mathbf{C}))_k$$

$$\mathcal{E}_{ijk} A_j (\mathbf{B} \times \mathbf{C})_k = \mathcal{E}_{ijk} A_j \mathcal{E}_{klm} B_l C_m$$

i, j, k, l and m are called dummy variables which can be equal to 1, 2 or 3 at any time i.e.,

$$i, j, k = 1, 2, 3, \text{ and}$$

$$k, l, m = 1, 2, 3$$

We continue to follow the convention that we must sum over repeated indices.

At this point we can incorporate one of the identities from above, so that we now have

$$\begin{aligned} \mathcal{E}_{ijk} A_j \mathcal{E}_{klm} B_l C_m &= (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) A_j B_l C_m \\ &= \delta_{il} \delta_{jm} A_j B_l C_m - \delta_{jl} \delta_{im} A_j B_l C_m \\ &\quad (I) \qquad (II) \end{aligned}$$

We can now expand each of these terms on the right to see if there are any terms that can be excluded from the derivation.

There are too many terms to try to do this in a brute force way, so let's first ask **four** big-picture questions which cover all four possibilities.

Q1. In so doing, we should think in which cases the combination of indices will provide either $I=0$ OR $II=0$?

i.e., when will $I=0$?..... OR when will $II=0$?.....

For the case of the first term on the right,

$$I=0 ? \quad \text{when } i \neq l \mid j \neq m$$

$$II=0 ? \quad \text{when } j \neq l \mid i \neq m$$

Q.2 Let us ask a second question before we get confused by the terms.

In which cases, will both terms on the right be equal, i.e.,

$$I=II ? \quad \text{when} \\ i=l \ \& \ j=m \ \& \ j=l \ \& \ i=m,$$

i.e., $i=l=j=m$, when all the indices are simultaneously equal.

For this case $I-II=0$ (!!), whether they are individually equal to 0 or not.

Q.3 Finally, for which case will $I=0$ AND $II=0$?

$$I=II=0? \quad \text{when } i \neq l \mid j \neq m \text{ AND } j \neq l \mid i \neq m$$

Q. 4 Now, the only cases that will be left, will be those where

$$I \neq 0 \text{ AND } II \neq 0, \text{ (both are non-zero)}$$

$$I \neq 0 ? \quad \text{when } i=l \ \& \ j=m$$

$$II \neq 0 ? \quad \text{when } j=l \ \& \ i=m$$

$$I \neq 0 \text{ OR } II \neq 0 \text{ (i.e., only one of the two terms is non-zero)}$$

Now thanks to this overview, we know that the only cases which will contribute are those where either terms on the right are not equal to 0 or to each other.

So, when is $I \neq 0$ and $i = l$ & $j = m$ we obtain the following expression:

$$\begin{aligned}
 \mathcal{E}_{lmk} \mathbf{A}_m \mathcal{E}_{klm} \mathbf{B}_l \mathbf{C}_m &= (\delta_{ll} \delta_{mm} - \delta_{ml} \delta_{lm}) \mathbf{A}_m \mathbf{B}_l \mathbf{C}_m \\
 (\mathcal{E}_{lmk}^2 \mathbf{A}_m \mathbf{B}_l \mathbf{C}_m) &= \delta_{ll} \delta_{mm} \mathbf{A}_m \mathbf{B}_l \mathbf{C}_m - \cancel{\delta_{ml} \delta_{lm} \mathbf{A}_m \mathbf{B}_l \mathbf{C}_m} \\
 (\mathbf{A}_m \mathbf{B}_l \mathbf{C}_m) &= \mathbf{A}_m \mathbf{B}_l \mathbf{C}_m \\
 &= \mathbf{A}_m \mathbf{C}_m \mathbf{B}_l \\
 &= (\mathbf{A} \cdot \mathbf{C}) \mathbf{B}_l \\
 &\quad (I)
 \end{aligned}$$

(Recall from earlier in these notes that the scalar product in indicial notation is

$$\mathbf{V}_i \mathbf{W}_j \delta_{ij} = |\vec{V} \cdot \vec{W}|$$

So, when is $II \neq 0$ and $j = l$ & $i = m$ we obtain the following expression:

$$\begin{aligned}
 \mathcal{E}_{mlk} \mathbf{A}_l \mathcal{E}_{klm} \mathbf{B}_l \mathbf{C}_m &= (\delta_{ml} \delta_{lm} - \delta_{ll} \delta_{mm}) \mathbf{A}_l \mathbf{B}_l \mathbf{C}_m \\
 &= \cancel{\delta_{ml} \delta_{lm} \mathbf{A}_l \mathbf{B}_l \mathbf{C}_m} - \delta_{ll} \delta_{mm} \mathbf{A}_l \mathbf{B}_l \mathbf{C}_m \\
 &= -\delta_{ll} \delta_{mm} \mathbf{A}_l \mathbf{B}_l \mathbf{C}_m \\
 &= (-\delta_{ll} \delta_{mm} \mathbf{A}_l \mathbf{B}) \mathbf{C}_m \\
 &= -\mathbf{A}_l \mathbf{B}_l \mathbf{C}_m \\
 &= -(\mathbf{A} \cdot \mathbf{B}) \mathbf{C}_m \\
 &\quad (II)
 \end{aligned}$$

(QED)

General Matrix Multiplication

In general matrix multiplication, we multiply over the columns and add over the rows. Different to dot and cross products of vectors we can multiply matrices of variable dimensions. The only restriction is that the number of columns (j) in the first matrix is equal to the number of rows (i) in the second matrix during the multiplication.

Using indicial notation, for two different matrices, \mathbf{A} and \mathbf{B} ,

$$C_{ij} = A_{ik} B_{kj},$$

Where, for matrix A , i is the number of rows and k the number of columns. For matrix B , i is the number of rows and k the number of columns. For matrix C , i is the number of rows and k the number of columns.

In order to be able to multiply matrices, the number of columns in the first matrix must equal the number of rows in the second matrix (Hence k is repeated). Note that i and j can be different and that in the following example we are assuming a special case where they are equal to 3.

According to indicial notation, since the k index is repeated then there must be a summation between the multiplications of columns of the first matrix and rows of the second. So, in a more expanded form we obtain:

$$C_{ij} = \sum_{k=1}^3 A_{ik} B_{kj}$$

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j}$$

If we now consider every possible combination of i and j , where $i=1,2,3$ and $j=1,2,3$ we have 9 permutations (tensor rank=3 (=dimension), order=2)

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ C_{13} &= A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\ C_{23} &= A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\ C_{31} &= A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} \\ C_{32} &= A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \\ C_{33} &= A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \end{aligned}$$

These permutations can be placed into a matrix, consisting of 3 rows and 3 columns

$$C_{ij} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

In MatlabTM:

```
>> A= [1 0 1;0 0 0]
```

```
A = 1    0    1
```

0 0 0

```
>> B= [2 0 3;1 3 3;1 2 2]
```

B =

2 0 3

1 3 3

1 2 2

```
>> A*B
```

ans = 3 2 5

0 0 0

Q. Do this example above by hand

References

- Maxwell, S. C., Rutledge, J., Jones, R., and Fehler, M., 2010, Petroleum reservoir characterization using downhole microseismic monitoring: *Geophysics*, v. 75, no. 5, p. 75A129-175A137.
- Wielandt, E., 2009, Seismic sensors and their calibration, *in* Bormann, P., ed., *New Manual of Seismological Observatory Practice*, Volume 1: Potsdam, GeoForschungsZentrum.