Content

DIVERGENCE

GRADIENT

CURL

DIVERGENCE THEOREM

LAPLACIAN

HELMHOLTZ'S THEOREM

DIVERGENCE

Divergence of a vector field is a scalar operation that in once view tells us whether flow lines in the field are parallel or not, hence "diverge".

For example, in a flow of gas through a pipe without loss of volume the flow lines remain parallel, but if the pipe narrows and the gas experiences compression then the flow lines in the gas will converge (i.e. divergence is not zero)

Another term for the divergence operator is the 'del vector', 'div' or 'gradient operator' (for scalar fields). The divergence operator acts on a vector field and produces a scalar. In contrast, the gradient acts on a scalar field to produce a vector field.

When the divergence operator acts on a vector field it produces a scalar. In contrast, the gradient operator acts on a scalar field to produce a vector field.

The divergence vector operator is ∇ (also known as 'del' operator) and is defined as

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix}, \text{ or, } \vec{\nabla} = \hat{x}_1 \frac{\partial}{\partial x_1} + \hat{x}_2 \frac{\partial}{\partial x_2} + \hat{x}_3 \frac{\partial}{\partial x_3}$$

in either indicial notation, or Einstein notation as

$$\vec{\nabla} = \nabla = \frac{\partial}{\partial x_i} x_i = ,_i$$

For example,

$$\nabla \cdot \boldsymbol{V} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{V} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}, \text{ or,}$$

in indicial notation, as

$$\nabla \boldsymbol{V} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

$$V_{i,i} = a_{i,i}$$

We will see more on indicial notation later but note that the summation is implied by repeated indices *(ii)* and that a "*,i*" denotes derivative with respect to the variable *i (*=1,2,3*)*

(Note that I have omitted the dot; the dot is not essential, but like a 'dot product' of two vectors the outcome is a scalar value.)

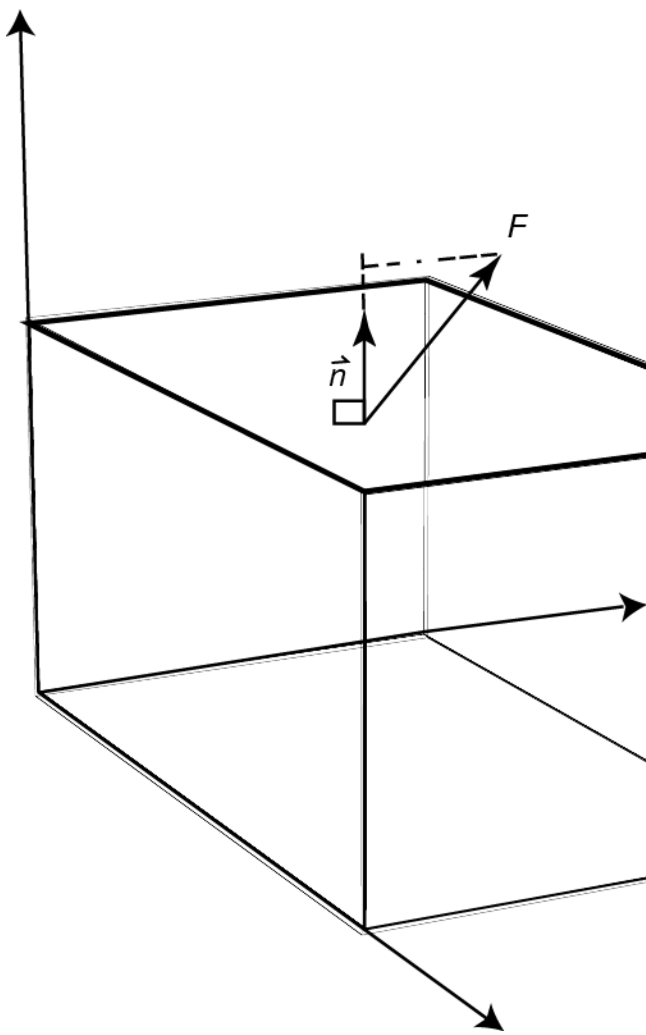
(Note that I have omitted the dot; the dot is not essential—it represents abuse of mathematical notation, although it is still correct)

Schey p. 36-40 "Div, Grad and All That"

$$\nabla \cdot \boldsymbol{V} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{V} = \left(\frac{\partial a_1}{\partial x_1} \quad \frac{\partial a_2}{\partial x_2} \quad \frac{\partial a_3}{\partial x_3} \right), \text{ or,}$$

If we consider the physical meaning of a divergence. Start with a cube and estimate the surface integral of some function such as the force applied at all points on its surface:



$\vec{n}$ is the unit vector normal to the faces of a cube

dS is the area of each face of the cube

(x_1, x_2, x_3) is a point at the centre of the cube. We are interested in the ratio of the integral of this function to the volume enclosed by the cube. This limit of this ratio is called the divergence:

$$\nabla \cdot \vec{F} = "divF" = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint \vec{F} \cdot \vec{n} dS$$

(Gauss' Theorem)

$\left(\nabla \cdot \vec{F}\right)$ is a scalar.

If, for example we examine the divergence of the electrostatic field, then the sum of the field over the faces can give us an idea of the charge included in the volume. If we sum the flow of the field over the faces and ad the total flow on all 6 faces and then make the cube diminish to a very small size, then we should be able to determine the flux at the point.

If we measure the sum of all the displacements/strains around the face of the cube and in the limit they tend to zero then we can say that there is no net change in the volume of the object. That would mean that as much strain is moving the faces in one direction as there are net strains compensating for deformation on the other side of the cube.

For cases of pure and simple shear there is no net change in area of volume, i.e. volume is conserved. In this situation,

$$\nabla \cdot \vec{u} = 0$$

$\nabla \cdot \vec{u}$ can also be viewed as the volumetric strain:

$$\nabla \cdot \vec{u} = \frac{\nabla V}{V}$$

$$\nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

You will see later that the strain tensor is defined in general as:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The strain tensor is equal to the divergence of the displacement field, when $i=j$.

Show that the divergence is the volumetric strain by the geometry of a deforming cube. Assume that the cube is growing only in the direction of the basis vectors.

GRADIENT

Gradient is the derivative of a scalar field and is also known as the "grad".

The gradient of a scalar field produces a vector. The gradient is written as:

$$\nabla = \frac{\partial}{\partial x_1}\hat{x}_1 + \frac{\partial}{\partial x_2}\hat{x}_2 + \frac{\partial}{\partial x_3}\hat{x}_3$$

The greatest spatial rate of change occurs in the direction of the gradient.

The gradient is invariant to the coordinate transformation:

For a simple 2D example:

$$\nabla h(x,y) = \frac{\partial h(x,y)}{\partial x}\hat{x} + \frac{\partial h(x,y)}{\partial y}\hat{y}$$

In indicial notation, the above example becomes:

$$\nabla h(x,y) = \frac{\partial h(x_i)}{\partial x_i}\hat{x}_i$$

The magnitude of this vector in the two-dimensional case is:

$$|\nabla h| = \sqrt{\left(\frac{\partial h(x,y)}{\partial x}\right)^2 + \left(\frac{\partial h(x,y)}{\partial y}\right)^2}$$

CURL

The "curl" or "rot" of a vector field is defined as

$$\nabla \times \vec{u} = \begin{pmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{pmatrix}$$

In indicial notation, this can be written as:

$$\begin{aligned} f_i &= (\nabla \times \vec{u})_i \\ &= \boldsymbol{\varepsilon}_{ijk} \nabla_j \boldsymbol{u}_k \end{aligned}$$

$f_i = \boldsymbol{\varepsilon}_{ijk} \boldsymbol{u}_{k,j}$, or in mixed notation:

$$f_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

For clarification, we can write out the individual components long-hand:

$$f_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}$$, because if i=1, j=2 or 3 alone for non-zero results

Similarly,

$$f_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}$$

$$f_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$$

DIVERGENCE THEOREM AND THE LAPLACIAN

Gauss's Theorem (or divergence theorem) states that the flux of a property over the surface of a volume equals the divergence of the property added up over the whole volume enclosed by the same surface. The integral of the divergence over the volume tells use whether that property is changing in size. That is,

$$\int_V \nabla \cdot \vec{F} dV = \int_S \vec{F} \cdot \vec{n} dS$$

Where $\vec{n}$ is the vector normal to the surface at any point and $\vec{F}$ is the vector field property in question.

If we make:

$\vec{F} = \nabla u$, then

$$\int_V \nabla \cdot \nabla u dV = \int_S \nabla u \cdot \vec{n} dS ,$$

$$\int_V \nabla^2 u dV = \int_S \nabla u \cdot \vec{n} dS$$ or

$$\int_V \Delta u dV = \int_S \nabla u \cdot \vec{n} dS$$

"The sum of the Laplacian (for a scalar field) over the volume is also the sum of the gradient over the entire surface"

Note that we have just introduced the Laplacian operator.

LAPLACIAN

(From Lay and Wallace, 1995)

When the Laplacian operator acts on a scalar field (u) it is equivalent to taking first the gradient followed by the divergence of the result, i.e.:

$$\text{Laplacian} = \text{“}\nabla \cdot \nabla\text{”}$$

$$\begin{aligned}\nabla^2 u &= u_{,ii} \\ &= \nabla \bullet \left(\frac{\partial\, u(x_1,x_2,x_3)}{\partial x_1}, \frac{\partial\, u(x_1,x_2,x_3)}{\partial x_2}, \frac{\partial\, u(x_1,x_2,x_3)}{\partial x_3} \right) \\ &= \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}, \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}, \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\end{aligned}$$

The Laplacian can also operate on a vector field ($\vec{F}$), in which case it is equivalent to

another vector field whose components are the Laplacian of the original vector components (if Cartesian coordinates are used)

$$\begin{aligned}\left(\nabla^2 \vec{F}\right)_i &= \vec{F}_{i,i}\hat{x}_i \\ &= \frac{\partial^2 \boldsymbol{F}_1}{\partial x_1^2}\hat{x}_1 + \frac{\partial^2 \boldsymbol{F}_2}{\partial x_2^2}\hat{x}_2 + \frac{\partial^2 \boldsymbol{F}_3}{\partial x_3^2}\hat{x}_3\end{aligned}$$

In the general case for any coordinate system we can use the following vector identity, seen elsewhere (->)

HELMHOLTZ'S THEOREM

(From Lay and Wallace, 1995)

This theorem states that any vector field $\vec{F}$, can be represented in terms of a vector potential ($\vec{\Omega}$) and a scalar potential (Θ) by

$$\vec{F} = \nabla\Theta + \nabla \times \vec{\Omega}$$

if $\nabla \times (\nabla\Theta) = 0$ and $\nabla \cdot (\nabla \times \vec{\Omega}) = 0$

which are useful vector identities are:

$$\nabla \cdot (\nabla \times \vec{\Omega}) = 0$$ (div of curl of vector field)

$$\nabla \times \left(\nabla \Theta\right) = 0$$ (curl of grad of scalar field)