Modeling groundwater–surface water interactions using the Dupuit approximation

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Abstract

Global errors in head and/or discharge may be introduced when groundwater flow to a stream is modeled using the Dupuit approximation. We consider a simple case of steady groundwater flow in the vertical plane to a horizontal stream bed in direct connection with the aquifer, and compare solutions to the exact problem with Dupuit solutions where common representations of the stream are chosen. In all cases considered, adopting the Dupuit approximation introduces global errors into the mathematical model, and the magnitude of the errors depends on the regional flow conditions. This behavior makes calibration of a model difficult and limits the predictive abilities of the model under conditions of changed regional flow. The global errors and their dependence on flow conditions can be minimized, but not eliminated by treating the resistance of a fictitious leaky stream bed as an effective parameter.

We propose an alternate Dupuit model of groundwater–surface water interaction and demonstrate, for the case considered, that adding a second effective parameter allows us to eliminate global errors in head and discharge, and eliminate the dependence of the effective values on the flow field. Explicit expressions are provided to evaluate the two effective properties. We propose that the results be used as a general guideline for modeling groundwater–surface water interaction at streams.

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1. Introduction

Surface water features play a prominent role in nearly every model of groundwater flow as they represent real hydrologic boundaries that control the behavior of the regional flow field. They also present difficulties for modelers. As discussed by Hunt et al. [11], the manner in which surface water features are represented in a groundwater model can have significant effects on the model results. Hunt et al. [11] investigated and compared alternate conceptual models for simulating groundwater–lake interaction and concluded that the potential for developing poor models of groundwater–surface water interaction is quite high; they suggest that this may be a result of the availability of increasingly sophisticated numerical packages, that can easily be applied without insight or understanding.

It is recognized that stream–aquifer or lake–aquifer interactions create complex flow patterns that include a variety of possible flow regimes linking the local groundwater flow field to the regional flow [19,16,2]. Today, the most common approach for analyzing the interaction of groundwater and surface water is numerical modeling; many modeling studies focus on the identification of flow regimes based on the location of stagnation points in the flow field [21,20,5,15]. In regional models, the Dupuit approximation is often made, and the full details of groundwater–surface water interaction at streams are...
difficult to include due to the computational effort required for fine vertical discretization in an areally extensive domain. Global errors in head and discharge can be introduced into a model when employing the Dupuit approximation in regions of concentrated vertical flow [9,7,3], such as occur near surface water bodies. These global errors in the Dupuit model can result in biased estimates of aquifer properties when calibrating a model to observed head data. Kaleris[12] demonstrated, using a Dupuit model, that forecasting groundwater–surface water interaction under flow conditions other than those used for calibration will give erroneous results; this is especially true in cases where the resistance of the stream bed is small and the stream is narrow. As surface water bodies are a common feature of nearly all groundwater flow models, this limitation restricts the predictive capabilities of Dupuit-based flow models.

In many cases, Dupuit models can be improved by defining effective aquifer properties that incorporate, in an approximate fashion, the head losses due to vertical flow that are neglected in the Dupuit formulation: Streltova[18] defined an additional seepage resistance to deal with vertical flow for several groundwater flow problems; Halec and Svec[10] described the method of substitute lengths to approximate losses due to vertical recharge from a reservoir; Strack[17] used an equivalent length of aquifer to include losses due to vertical flow to a partially penetrating ditch; Bakker[8] defined an effective resistance to vertical flow in a two layer Dupuit model of flow to a ditch. In all these cases, an effective property is defined and evaluated by comparing a Dupuit solution to an exact (vertical plane) solution. All examples above represent analyses of small scale engineering problems; the procedures do not fit well within the framework of regional groundwater flow modeling.

Here, we investigate global errors that are introduced into regional models by standard applications of the Dupuit approximation, and develop a Dupuit model that eliminates these global errors. Our primary objective is to develop simple techniques to improve regional models in which surface water features appear as boundaries without increasing the computational burden associated with either local or global vertical discretization. We achieve this objective by defining and evaluating effective properties that are independent of the flow conditions in the aquifer and fit well into standard modeling practices. Our secondary goal is to examine the extent to which a Dupuit model, assigned appropriate effective properties, can distinguish between local flow regimes and make accurate predictions of the local flow field.

2. Common Dupuit representations of streams

We consider an idealized problem of groundwater flow to examine and quantify global errors that are introduced when adopting the Dupuit approximation. We consider steady groundwater flow to a stream in a vertical section, as shown in Fig. 1. The aquifer is an infinite strip of thickness \( H \) [L] with an impermeable bottom and top, except along the stream bed of width \( D \) [L], shown as the dashed line; the hydraulic head is constant and equal to \( h^* \) [L] along the stream bed and the stream is in direct connection with the aquifer. Discharges \( Q_l \) and \( Q_r \) \([L^2/T]\) are specified at the left and right sides of the domain, respectively, and the aquifer has a hydraulic conductivity of \( k \) \([L/T]\).

Fig. 2a and b show two representations of the stream used commonly in regional groundwater flow models. In Fig. 2a, the resistance to vertical flow of the aquifer is entirely neglected by assuming full penetration of the stream. As a result, the stream is modeled by fixing

![Fig. 1. Definition sketch—Groundwater–surface water interaction in the vertical plane.](image)

![Fig. 2. Common Dupuit representations of the stream: (a) the 0P model with a fully penetrating stream; (b) the 1P model with resistance to vertical flow incorporated approximately into the solution by including a fictitious leaky stream bed; and (c) the 2P model where a heterogeneity of hydraulic conductivity \( k' \) is introduced below the stream.](image)
the head \( h^* \) at both edges of the stream. We refer to this conceptual model as the zero-parameter (0P) Dupuit model. In Fig. 2b, the flow beneath the stream is modeled as semi-confined flow, being separated from the stream by a fictitious leaky stream bed of resistance \( c \) [T]. The resistance of the stream bed is used in the Dupuit model to approximate the resistance to vertical flow of the aquifer. We refer to this representation of the stream as the one-parameter (1P) Dupuit model.

In Fig. 2c we introduce an alternate Dupuit representation of the stream by including an additional parameter, \( k^* \) [L/T], to represent the hydraulic conductivity of the aquifer beneath the stream. We refer to this new conceptual model as the two-parameter Dupuit (2P) model. We include this second parameter, \( k^* \), now for generality, making use of it later; in a standard application, \( k^* \) equals \( k \), the hydraulic conductivity of the aquifer, and the 2P model reduces to the 1P model.

In the following sections, we present analytical solutions to both the exact problem and the three alternate Dupuit representations of the same problem. We will use the solutions to examine errors in the Dupuit model.

3. Analytic solutions

3.1. The exact problem

The solution to the exact problem, illustrated in Fig. 1, was developed originally by Aravin and Numerov [6] and may be obtained by conformal mapping and the method of images. We define a complex potential, \( \Omega \) [L²/T], that is an analytic function of the complex coordinate \( z = x + iy \)

\[
\Omega = \phi + i \psi \quad (1)
\]

where \( \psi \) is the stream function, \( \phi \) is the specific discharge potential

\[
\phi = kh \quad (2)
\]

and where \( h \) is the hydraulic head in the aquifer. We also define the complex specific discharge, \( W \) [L/T], as

\[
W = -\frac{\partial \Omega}{\partial z} = q_x - i q_y \quad (3)
\]

where \( q_x \) and \( q_y \) are the \( x \)- and \( y \)-components of the specific discharge vector, respectively.

A uniform specific discharge of strength \( Q_l/H \) is specified at \( x = -\infty \); at \( x = +\infty \) a uniform specific discharge of \( Q_r/H \) is specified. The boundary conditions for the exact problem follow:

\[
\Im(\Omega) = -Q_\ell; \quad -\infty \leq x \leq 0, \quad y = 0 \quad (4)
\]

\[
\Re(\Omega) = kh^*; \quad 0 \leq x \leq D, \quad y = 0 \quad (5)
\]

\[
\Im(\Omega) = -Q_r; \quad D \leq x \leq +\infty, \quad y = 0 \quad (6)
\]

The complex potential satisfying these conditions may be written in the following form [2]:

\[
\Omega = \frac{Q_\ell}{\pi} \ln \left[ \frac{\zeta^{1/2} - 1}{\zeta^{1/2} + 1} \right] - \frac{Q_r}{\pi} \ln \left[ \frac{\zeta^{1/2} - \gamma^{1/2}}{\zeta^{1/2} + \gamma^{1/2}} \right] + kh^* + iQ_t \quad (8)
\]

where

\[
\zeta = \frac{\gamma \exp[-\pi(z-D)/H] - 1}{\exp[-\pi(z-D)/H] - 1} \quad (9)
\]

and

\[
\gamma = \exp \left[ -\frac{\pi D}{H} \right] \quad (10)
\]

We wish to develop expressions for the head in the aquifer a distance \( L \) away from the edges of the stream

\[
h_1 = h(z = -L) \quad (11)
\]

\[
h_2 = h(z = L + D) \quad (12)
\]

To evaluate (11) for large values of \( L \), we first expand (9) about \( z = -\infty \). From (9) we find

\[
\lim_{z \to -\infty} \zeta^{1/2} = \gamma^{1/2} \quad (13)
\]

Expanding \( \zeta^{1/2} \) about \( z = -\infty \), we obtain

\[
\zeta^{1/2} = \gamma^{1/2} + \frac{(\gamma - 1)}{2\gamma^{1/2}} \epsilon + O(\epsilon^2) \quad (14)
\]

where

\[
\epsilon = \exp \left[ \frac{\pi}{H} (z-D) \right] \quad (15)
\]

or

\[
\zeta^{1/2} - \gamma^{1/2} \approx \frac{(\gamma - 1)}{2\gamma^{1/2}} \exp \left[ \frac{\pi}{H} (z-D) \right] \quad (16)
\]

which is valid for large negative values of \( z \). We specify \( z = -L \) and substitute (16) into (8) for the term \( \zeta^{1/2} - \gamma^{1/2} \) and set \( \zeta^{1/2} = \gamma^{1/2} \) in the remaining terms. Taking the real part of the resulting expression and using (1), (2) and (10) we obtain

\[
h_1 \approx \frac{Q_L}{kh} + \frac{Q_{\ell}}{k\pi} \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] - \frac{Q_r}{k\pi} \ln \left[ \frac{1}{4} (1 - \gamma) \right] + h^* \quad (17)
\]

\[
\left( \text{for } \frac{L}{H} \gg 1 \right)
\]

We follow a similar approach to evaluate \( h_2 \): we expand (9) about \( z = +\infty \) (\( \zeta^{1/2} = 1 \)) and evaluate (8) at \( z = L + D \) to obtain

\[
h_2 \approx -\frac{Q_L}{kh} + \frac{Q_r}{k\pi} \ln \left[ \frac{1}{4} (1 - \gamma) \right] - \frac{Q_{\ell}}{k\pi} \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] + h^* \quad (18)
\]

\[
\left( \text{for } \frac{L}{H} \gg 1 \right)
\]
In both expressions, (17) and (18), the first term on the right-hand side of the equations represents the effects of one-dimensional uniform flow on the distribution of head in the aquifer, and the second and third terms reflect exactly the additional head loss on either side of the stream, due to non-uniform flow throughout the aquifer. The expressions, (17) and (18), are both developed with asymptotic expansions and are therefore valid only for large values of $L/H$. However, we find that $L/H \geq 1.5$ is adequate to produce accurate results. This may be seen by evaluating $\epsilon$ (15) for $z = -1.5H$ and any value of $D$. The leading truncated term in the expansion (14) is of the order $\epsilon^2$, which has a maximum value of $8 \times 10^{-5}$.

3.2. The Dupuit problem

We solve the two-parameter Dupuit problem as illustrated in Fig. 2c; the 0P and 1P models will then be solved as special cases of the 2P solution. The domain for the Dupuit problem is split into three parts, as illustrated in the figure. We define a discharge potential $\Phi$ [L$^2$/T] for each zone

$$\Phi = \begin{cases} kH(h-h^*), & \text{for } x \leq 0 \\ k^2H(h-h^*), & \text{for } 0 \leq x \leq D \\ kH(h-h^*), & \text{for } x \geq D \end{cases} \tag{19}$$

where $H$ is the aquifer thickness. The governing differential equation in each zone is

$$\nabla^2 \Phi = \begin{cases} 0, & \text{for } x \leq 0 \\ \Phi/\lambda^2, & \text{for } 0 \leq x \leq D \\ 0, & \text{for } x \geq D \end{cases} \tag{20}$$

where $\lambda$ [L] is the leakage factor defined as

$$\lambda = \sqrt{k^2\varepsilon c} \tag{21}$$

The x-component of the discharge vector (the depth integrated specific discharge vector), $Q_x$ [L$^2$/T], may be obtained from the discharge potential as

$$Q_x = -\frac{d\Phi}{dx} \tag{22}$$

The general solution for one-dimensional flow in each zone is (e.g. [17])

$$\Phi = \begin{cases} -Q_x x + C_1, & \text{for } x \leq 0 \\ A \exp(x/\lambda) + B \exp(-x/\lambda), & \text{for } 0 \leq x \leq D \\ -Q_x(x-D) + C_2, & \text{for } x \geq D \end{cases} \tag{23}$$

where $C_1$, $A$, $B$, and $C_2$ are constants that may be evaluated by applying continuity of head and discharge across the two boundaries ($x = 0$ and $x = D$) joining the three zones. The constants are evaluated as

$$C_1 = -\frac{k}{k^*} \left[ \frac{Q_x - Q_o \cosh(D/\lambda)}{\sinh(D/\lambda)} \right] \tag{24}$$

$$A = \frac{1}{2} \left( \frac{k^*}{k} C_1 - \lambda Q_1 \right) \tag{25}$$

$$B = \frac{1}{2} \left( \frac{k^*}{k} C_1 + \lambda Q_1 \right) \tag{26}$$

$$C_2 = C_1 \cosh \left( \frac{D}{\lambda} \right) - \lambda Q_1 \frac{k}{k^*} \sinh \left( \frac{D}{\lambda} \right) \tag{27}$$

The head to the left and right of the stream may be evaluated using (19), (23), (24), and (27)

$$h_l = h(x = -L) = \frac{Q_1 L}{kH} + C_1 + h^* \tag{28}$$

$$h_r = h(x = L + D) = -\frac{Q_1 L}{kH} + C_2 + h^* \tag{29}$$

The first term on the right-hand side of each expression represents the effects of the uniform flow; those terms are identical to the first terms in (17) and (18), respectively. The second terms in (28) and (29) reflect the additional head loss induced by the resistance of the fictitious, leaky stream bed.

4. Errors generated by common Dupuit representations of the stream

We evaluate errors between solutions of the exact and the 0P and 1P Dupuit problems when common values for the resistance, $c$, of the fictitious leaky stream bed are chosen. There are a variety of ways to measure either errors in discharge or errors in heads between models, depending on how boundary conditions are specified. We choose to specify the same regional discharges ($Q_0$ and $Q_1$) in the exact and Dupuit models and evaluate the difference in head a short distance from the stream. The difference between the exact head and Dupuit head will approach a constant value as we move a small distance from the stream; this is the global error introduced into the model by adopting the Dupuit approximation. We define the error in the Dupuit model, using (18), (29), and (17) as

$$\epsilon(\%) = \frac{(h_l)_{\text{Dupuit}} - (h_l)_{\text{Exact}}}{(h_l - h^*)_{\text{Exact}}} \times 100 \tag{30}$$

In calculating the error, we maintain $(h_l)_{\text{Exact}}$ as a fixed value and vary $Q_x/Q_l$ from minus one to plus one. This procedure guarantees that the following condition is met:

$$(h_l - h^*)_{\text{Exact}} \leq |(h_l - h^*)_{\text{Exact}}| \tag{31}$$

For each value of $Q_x/Q_l$ we compute $(h_l)_{\text{Exact}}$ and the individual values of $Q_x$ and $Q_l$. We then specify these
values of discharge in the Dupuit model and calculate \((h_2)_{\text{Dupuit}}\).

In (30) we normalize the error in head to the right of the stream between the Dupuit and exact models with the term \((h_1 - h_2)_{\text{Exact}}\). The denominator is a measure of the total head change across the local model and in all calculations it remains a constant positive value. We evaluate \(h_1\) and \(h_2\) at a distance \(L = 2H\) from the edge of the stream; as discussed earlier, the flow in the exact model becomes nearly uniform and the heads nearly constant over the vertical for \(L/H \geq 1.5\), making a comparison with the Dupuit models meaningful.

The errors are a function of the geometric parameter, \(D/H\), the regional flow parameter, \(Q_r/Q_1\), and the dimensionless leakage factor, \(\lambda H\). We need only consider the range \(-1 \leq Q_r/Q_1 \leq +1\) in our error calculations. Flow conditions outside that range may be evaluated by considering the symmetry of the problem domain about the stream centerline; for example, the behavior of the problem solution when \(Q_r/Q_1 = 5\) is the same as the case where \(Q_r/Q_1 = 1/S\), when the left and right sides of the domain are reversed.

### 4.1. Neglecting the vertical resistance to flow

We first consider the common case where the resistance of the stream bed is specified to be zero; this corresponds to the conceptual model of a fully penetrating stream, or the 0P model, as shown in Fig. 2a. For zero resistance, we find \(\lambda = 0\), \(C_1 = 0\), and \(C_2 = 0\) (see (21), (24), and (27)). Errors induced by assuming full penetration of the stream are presented in Fig. 3a. The error varies over the range of the flow parameter from about \(-18\%\) to \(+18\%\) for \(D/H = 3\); for streams of vanishing width the errors range from \(-61\%\) to \(0\%). The errors are large and vary strongly with regional flow conditions.

### 4.2. Vertical resistance as an intrinsic aquifer property

In the one-parameter Dupuit formulation shown in Fig. 2b, there is little guidance for selecting an appropriate value of resistance for the fictitious leaky stream bed. In numerical modeling, block-centered finite difference schemes treat the resistance between cells as an intrinsic aquifer property [3] reflecting the one-dimensional resistance between centers of vertically adjacent cells [13, pp. 5–13]. The resistance between vertically adjacent cells is given by

\[
c = \frac{H_1/2}{k_1} + \frac{H_2/2}{k_2} \tag{32}\]

where \(H_1\) and \(k_1\) are the depth and vertical hydraulic conductivity of the lower cell and \(H_2\) and \(k_2\) the depth and hydraulic conductivity of the overlying cell. Applied here to a one-layer model of groundwater interaction with a stream in direct contact with the aquifer, cell 1 represents the aquifer and cell 2 the overlying stream. The resulting intrinsic resistance of the stream bed for this case is evaluated by specifying \(k_1 = k^* = k, H_1 = H,\) and \(k_2 = \infty\) in (32) to obtain

\[
c = 0.5 \frac{H}{k} \tag{33}\]

From this resistance we obtain the leakage factor (21), made dimensionless by dividing by the aquifer thickness

\[
\frac{\lambda}{H} = \sqrt{\frac{ck}{kH}} = 0.707 \tag{34}\]

Specifying this intrinsic value for \(\lambda H\) in (28) and (29), yields the errors in head shown in Fig. 3b. For wide streams, the errors are reduced over the previous case and the variation with \(Q_r/Q_1\) is less; for \(D/H = 3\), errors range from \(-10\%\) to \(+12\%\). For \(D/H = 1\), the range of errors increases from \(-5\%\) to \(+22\%\). For narrow streams the error increases rapidly, exceeding \(100\%\) if the stream is narrower than \(0.5H\). Although specifying
the intrinsic value of resistance reduces both the magnitude of the error and variation with $Q_r/Q_l$ in most cases, the errors for very narrow streams are greater than when assuming full penetration of the stream. We also note that the contours of $Q_r/Q_l$ in Fig. 3b are inverted from those in Fig. 3a; the error associated with $Q_r/Q_l \geq 0$ are positive, whereas in Fig. 3a they are negative.

4.3. Vertical resistance as an effective property

It is common in modeling studies to treat some parameters as having effective values, using calibration to observed data to evaluate the parameter. Here we treat the resistance (or, equivalently, the leakage factor) of the fictitious leaky stream bed as an effective parameter in the 1P Dupuit model and assign a value of resistance that gives us a good match to the solution of the exact problem.

There are different measures of a good match—we choose the following for the purpose of illustration: we specify the regional discharges in the exact and Dupuit models and evaluate the effective resistance of the stream bed such that the head to the left of the stream in the Dupuit model (28) equals the head obtained from the exact solution (17).

To evaluate the effective resistance using this criteria, we set (17) equal to (28) and solve for the leakage factor, $\lambda$. Setting the two equations equal and specifying $k/k' = 1$, we obtain the following dimensionless equation:

$$Q_r = \frac{\cosh(D/\lambda) + (H/\lambda)(1/\pi) \ln \left[ \frac{1 - (1 - \gamma)}{1 + (1 - \gamma)} \right] \sinh(D/\lambda)}{1 + (H/\lambda)(1/\pi) \ln \left[ \frac{1 - (1 - \gamma)}{1 + (1 - \gamma)} \right]}$$

(35)

The dimensionless leakage factor $\lambda/H$ cannot be solved for explicitly. In terms of dimensionless parameters, we find that the effective value of the leakage factor, and thus the resistance, can be expressed as a function of the problem geometry ($D/H$) and the regional discharges ($Q_r/Q_l$). This relationship is shown graphically in Fig. 4, by contouring the expression (35). Use of the correct effective resistance, obtained from Fig. 4, allows the Dupuit model to reproduce the heads to the left of the stream exactly, although some error in heads to the right of the stream will occur. A sampling of the errors in head to the right, using (30) for the case $Q_r/Q_l = 0$, shows that the errors are approximately $-1.5\%$ over the range of $D/H$ from 0 to 3; there are no global errors to the left, and the errors to the right are small. Because of symmetry about the stream centerline, the flow cases $Q_r/Q_l = \pm 1$ have no error to the right.

4.4. A limiting case of effective resistance

A special case of interest is the problem of flow to an infinitely wide stream, or lake shore. We can develop this solution as a limiting case where $D \to \infty$. To evaluate this limit, we begin by clearing the denominator in (35) and dividing both sides by $\cosh(D/\lambda)$ to obtain

$$\frac{Q_r}{Q_l} \left\{ 1 + \frac{(H/\lambda)(1/\pi) \ln \left[ \frac{1 - (1 - \gamma)}{1 + (1 - \gamma)} \right]}{1 + (H/\lambda)(1/\pi) \ln \left[ \frac{1 - (1 - \gamma)}{1 + (1 - \gamma)} \right]} \right\} = 1 + \frac{H}{\lambda} \frac{1}{\pi} \ln \left[ \frac{1}{4} (1 - \gamma) \right] \tanh \left( \frac{D}{\lambda} \right)$$

(36)

For a stream of infinite width we obtain $\gamma = 0$ (10), $\cosh(D/\lambda) \to \infty$, and $\tanh(D/\lambda) = 1$. Substitution into (36) yields

$$\frac{\lambda}{H} = \frac{2}{\pi} \ln(2) = 0.441$$

(37)

or in terms of resistance

$$\frac{ck}{H} = \left( \frac{\lambda}{H} \right)^2 = 0.195$$

(38)

The dimensionless effective resistance becomes a constant, no longer dependent on the problem geometry or the regional flow conditions. This is identical to the results of Anderson [3] who considered the effective resistance to vertical flow in a Dupuit model of a two layer, stratified aquifer.

4.5. Discussion of common modeling practices

We see from the results of the previous sections that errors are introduced into the mathematical model when we adopt the Dupuit approximation and neglect the vertical resistance to flow. In regions of concentrated vertical flow the errors can be significant. The errors are shown to be a function of the problem geometry and the regional flow field, which are represented in this problem by the parameters $D/H$ and $Q_r/Q_l$, respectively.

Incorporating a vertical resistance to flow in the one-parameter Dupuit model by lumping the resistance along the stream bed can decrease errors if an appropriate
value of the resistance is chosen. Treating the resistance as an intrinsic property of the aquifer reduces errors, except when the stream is very narrow ($D/H < 0.5$). However, as shown in Fig. 3, the error is still a function of the regional flow conditions; this limits our ability to calibrate a model under different stress conditions, which is essential if the model is to be used to make predictions under conditions of changed stresses.

By treating resistance in the 1P model as an effective property, or a calibration parameter, we can reduce errors further, although the appropriate effective values vary with the flow field, again limiting the predictive abilities of a model. We see from Fig. 4 that in all cases the dimensionless effective resistance is much smaller than the intrinsic value of 0.5 (33). Apparently, the intrinsic value overestimates the vertical resistance to flow of the aquifer; as a result, an erroneous vertical anisotropy is built into the Dupuit model when using the intrinsic value. We can evaluate the erroneous anisotropy ratio by a trial and error procedure, and find that when specifying $ck/H = 0.5$, we build in an anisotropy ratio that ranges from 2.6 to 10, depending on the geometry and flow conditions.

The concept of an effective resistance helps to explain the errors associated with treating the resistance as an intrinsic aquifer property. However, effective properties are of practical use only if the effective values are independent or only weakly dependent on the boundary conditions of the problem ($D/H$ and $Q_r/Q_l$). Inspection of Fig. 4 shows this to be the case when $D/H$ gets large, and the solution behaves like that of flow to a lake shore; there is a weak dependence on the flow conditions near $D/H = 3$. However, for narrow streams, the effective resistance in the 1P model varies strongly with the flow parameter; it appears that the effective resistance is too strongly dependent on the regional discharges to be of practical use in groundwater flow modeling. A more direct assessment can be made by evaluating the errors that occur when approximating the effective resistance as having a single, constant value over the full range of $D/H$ and $Q_r/Q_l$. We test the dependence of the error in the Dupuit model to variations in flow conditions when specifying the effective dimensionless resistance obtained for the limiting case of flow to a lake shore (38). Fig. 5 shows the error plot when the dimensionless effective resistance is specified as 0.195. The figure shows a good match between the exact and Dupuit heads on the left of the stream bed. Overall, the errors in the one-parameter model are significantly reduced by using a single value of effective resistance for all geometries and flow conditions.

5. An alternate Dupuit model of groundwater–surface water interaction

A single value of effective resistance is adequate to remove most of the error in the 1P mathematical model for a wide range of geometric ratios ($D/H$) and flow conditions ($Q_r/Q_l$). When the stream is very narrow, or when no error is acceptable, an alternate approach may be taken. We demonstrate in this section that it is possible to eliminate all global error in the Dupuit model, for the problem considered here, by adopting the two-parameter model. We introduce the parameter $k^*$, the hydraulic conductivity beneath the stream, as shown in Fig. 2c, to allow the Dupuit model to match the exact heads to both the left and right of the stream.

We treat both $\lambda$ and $k^*$ as effective parameters in the Dupuit model and evaluate them by requiring the heads both to the left and right of the stream (28) and (29)) to be the same as those obtained from the solution to the exact problem ((17) and (18)). We obtain one constraint on the two effective parameters by equating (28) and (17),

$$\frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) + \ln \left[ \frac{1}{4} (1 - \gamma) \right] \sinh \left( \frac{D}{\lambda} \right) = \frac{Q_r}{Q_l} \left\{ \frac{\lambda}{H} \frac{k}{k^*} \pi + \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] \sinh \left( \frac{D}{\lambda} \right) \right\}$$

A second condition is obtained by equating the exact and Dupuit heads on the right. From (29) and (18), we obtain
\[
\begin{align*}
\lambda \cdot \frac{k}{H^2} \pi + \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] \sinh \left( \frac{D}{\lambda} \right) \\
\quad = \frac{Q}{Q_l} \left\{ \ln \left[ \frac{1}{4} (1 - \gamma) \right] \sinh \left( \frac{D}{\lambda} \right) + \frac{\lambda}{H} \cdot \frac{k}{k^*} \pi \cosh \left( \frac{D}{\zeta} \right) \right\} 
\end{align*}
\]

Conditions (39) and (40) are a pair of non-linear algebraic equations defining the dimensionless effective parameters \( \lambda H \) and \( k/k^* \). Surprisingly, the system may be solved explicitly for the two parameters; we demonstrate this in Appendix A. The results are:

\[
\lambda \equiv \frac{D}{H} \left\{ \text{acosh} \left[ \frac{\ln \left( \frac{1}{2} (1 - \gamma) \right)}{\ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right]} \right] \right\}^{-1}
\]

\[
k^* = -\frac{H}{\lambda} \frac{1}{\pi} \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] \sinh \left( \frac{D}{\lambda} \right)
\]

These results were reported without derivation by Anderson [1]. The dimensionless effective parameters \( k/k^* \) and \( \lambda H \) are plotted in Fig. 6, along with the dimensionless resistance of the stream bed, obtained from the effective leakage factor as

\[
ck^* = k^* \left( \frac{k}{H} \right)^2
\]

Inspection of Eqs. (41) and (42) reveals that both parameters are functions only of the geometric parameter \( D/H \); apparently this combination of effective parameters \( k^* \) and \( \lambda \) eliminates any dependence on the regional flow parameter, \( Q_r/Q \). This is significant, as we can now use the Dupuit model to forecast exchanges under conditions other than those used to calibrate the model. We have developed a model that produces the correct heads and discharges a short distance from the stream, on both sides of the stream. The values of

the effective parameters are entirely independent of the regional flow conditions in the aquifer, and only weakly dependent on the geometry.

5.1. Comparison of flow fields

In Figs. 7–9 we compare the exact flow field and the Dupuit flow field based on the two-parameter model under differing flow conditions. For this simple example of groundwater–surface water interaction, there are three possible flow regimes, characterized by the location of the stagnation point [15,4]. Here we refer to these flow cases as Regimes I, II and III. Regime I flow is characterized by a stagnation point lying along the aquifer base; Regime II flow is characterized by the stagnation point lying along the impermeable aquifer top; Regime III flow is characterized by a stagnation point lying along the stream bed. Groundwater flow in each regime interacts differently with the stream, and identifying the

![Fig. 6. Effective values for the two-parameter Dupuit model.](a)

![Fig. 7. Comparison of Regime I flow fields for \( D/H = 1 \) and \( Q_r/Q_l = -0.5 \): (a) flow net for the exact problem; (b) flow net for the 2P Dupuit problem; (c) a comparison of heads from the exact and Dupuit solutions; and (d) a comparison of the stream functions from the exact and Dupuit solutions.](b)
proper regime is important for estimating water budgets and predicting contaminant movement. We have chosen one representative example of flow from each of the three regimes, for the case that $\frac{D}{H} = 1$, to demonstrate the independence of the effective parameters on the regional flow conditions, and to provide a visual comparison of the exact and Dupuit solutions.

A comparison of the exact and Dupuit models for the case of Regime I type flow where $\frac{Q_r}{Q_l} = 0.5$ is given in Fig. 7. We present the flow net from the exact solution in Fig. 7a and the Dupuit solution in Fig. 7b. In both, contours of head are shown as dashed lines and contours of the stream function are shown as solid lines. The stream function for the Dupuit solution is developed following Anderson [3]. We show in Fig. 7c the exact (solid) and Dupuit (dashed) head distributions overlaid, for comparison. We observe that the head distributions from the two models converge a short distance from either edge of the stream bed. In a similar fashion, Fig. 7d shows the exact and Dupuit stream functions overlaid; here, the contours of stream function converge a short distance from the stream. The convergence of the heads and stream function to the exact values indicates that all global errors have been eliminated from the Dupuit model.

A similar comparison is made in Fig. 8 for the same problem under different flow conditions. The stagnation point in this problem lies on the aquifer top to the right of the stream bed. This is an example of Regime II type flow, where the stream has a capture zone that does not extend through the full depth of the aquifer; in this case, a very small portion of the flow bypasses the stream. The regional flow conditions are given by $\frac{Q_r}{Q_l} = 0.05$. Again in Figs. 8c and d we see the heads and stream functions from the two models converge rapidly away from the stream. Finally, we provide in Fig. 9 an example of Regime III flow, often referred to as a flow-through case, where the stagnation point lies on the stream bed. In this example, $\frac{Q_r}{Q_l} = +0.5$. Again, no global errors in head or discharge are seen in the Dupuit model.
Further investigation of each of these flow regimes shows that the flow behavior directly beneath the stream is predicted reasonably well by the Dupuit model. In Fig. 7, even the location of the stagnation point is predicted well by the Dupuit model. In Fig. 8, the exact stagnation point lies to the right of the stream bed, while the Dupuit dividing streamline lies at the right corner of the stream bed; the Dupuit model cannot predict the dividing streamline to lie off of the stream bed. In Fig. 9 the dividing streamline is not shown; we have plotted a comparison of the dividing streamline for Regime III flow in Fig. 10. We see that the location of the stagnation point is predicted accurately by the Dupuit model, but the value of the stream function along the dividing streamline is not the same in both models; the exact model predicts more water exiting the aquifer at the left side of the stream and more water entering to the right than does the Dupuit model. As a result the portion of the aquifer interacting with the stream is deeper in the exact case.

5.2. Transition diagram

We have evaluated effective properties by requiring there to be no global errors in heads or discharges in the Dupuit model. The flow fields presented in Figs. 7–9 also indicate comparable results between the exact and Dupuit solutions beneath the stream. For example, a quantitative comparison of the distribution of leakage along the stream bed can be made directly from Figs. 7–9 by comparing the spacing of the exact and Dupuit stream functions on the stream bed. We study the local behavior further by comparing transition diagrams developed for the exact and Dupuit solutions. A transition diagram is a dimensionless plot on which lines separating the flow regimes are drawn (e.g., [4,15]).

As discussed, we distinguish between flow regimes by the location of the stagnation point. The transition condition separating Regimes I and II flow is that the stagnation point lies at the right infinite boundary of the domain. In both the exact and Dupuit models this condition is equivalent to the condition $Q_t = 0$; in both models the transition line separating Regimes I and II flow, plotted in Fig. 11, are identical. We distinguish between Regimes II and III in the exact model by applying the following condition:

$$W(z = D) = 0$$  \hspace{1cm} (44)

This requires that the stagnation point lies at the right corner of the stream bed. We apply this condition to the exact solution to develop the transition line. First we evaluate the complex discharge (3) for the exact model

$$W = -\frac{d\Omega}{d\zeta}$$

Differentiating (8) and (9) and using (45), we obtain

$$W = \frac{1}{(\gamma + 1)H_s^{1/2}}\left[Q_t(\zeta - \gamma) - Q_l\gamma^{1/2}(\zeta - 1)\right]$$  \hspace{1cm} (46)

Applying condition (44) we obtain

$$\frac{Q_t}{Q_l} = \exp\left(-\frac{\pi D}{2H}\right)$$  \hspace{1cm} (47)

which is an equation of the exact transition line separating Regimes II and III, plotted in Fig. 11.

In the Dupuit model, the condition equivalent to (44) may be written as

$$\Phi(x = D) = 0$$  \hspace{1cm} (48)

This ensures that the dividing streamline touches the edge of the stream by forcing the vertical component of specific discharge to be zero there; positive values of $\Phi(x = D)$ lead to Regime II type flow, while negative values lead to Regime III flow. From (23) for the zone to the right of the stream, and (27) and (24) we obtain

$$-\lambda \frac{k}{k} \left[\frac{Q_t - Q_l \cosh(D/\lambda)}{\sinh(D/\lambda)}\right] \cosh\left(\frac{D}{\lambda}\right) + Q_l \sinh\left(\frac{D}{\lambda}\right) = 0$$  \hspace{1cm} (49)

Simplifying gives

$$\frac{Q_t}{Q_l} = \frac{1}{\cosh(\frac{D}{\lambda})}$$  \hspace{1cm} (50)
We use the identity (55) derived in Appendix A to obtain an equation for the transition line for the Dupuit model, separating Regimes II and III:

\[ \frac{Q_l}{Q_t} = \ln \left[ \frac{1 - \gamma / \gamma_2}{\gamma_1 / \gamma_2} \right] \]  

The transition diagram is plotted in Fig. 11. The transition line between Regimes I and II are identical in both the exact and Dupuit models, as discussed earlier; the transition line between Regimes II and III are plotted and labeled on the transition diagram. There is some error in the Dupuit model, but the two curves are comparable over the full range of parameters.

6. Conclusions

Surface water features appear in nearly every model of groundwater flow as they represent real hydrologic boundaries. Regions of concentrated vertical flow often occur near surface water features, bringing the results of Dupuit models into question; adopting the Dupuit approximation introduces errors into the model results and although these errors may be small in comparison to errors due to uncertainty in model parameters, the errors are systematic. To improve a flow model of groundwater–water interacting with a surface water feature, it is often proposed that the model be discretized vertically to incorporate more accurately vertical flows and the associated head losses. In regional-scale numerical models this can add a significant computational burden, even when vertical discretization is performed only locally [14]. We have investigated a worst-case situation for the Dupuit model, where the stream is in direct contact with the aquifer, and found that vertical discretization is often unnecessary if the resistance to vertical flow is incorporated approximately into the model.

We have devised an analytical framework within which to study the accuracy of single-layer Dupuit models of groundwater–surface water interaction, and we have verified earlier observations of Bear and Braester [9], and Bakker [7] that use of the Dupuit approximation in regions of concentrated vertical flow leads to local and global errors in heads and/or discharges. We found that neglecting the resistance to vertical flow entirely (0P model) or treating it as an intrinsic aquifer property (1P model) leads to significant global errors that vary with changes in the regional flow conditions. This variation with flow conditions makes calibration of a model using different stresses difficult, and limits the ability of the model to make accurate predictions under changed stresses.

The global errors introduced by adopting the Dupuit approximation can be reduced significantly by treating the resistance of the stream bed in the 1P model as an effective property. In the case of a stream in direct contact with the aquifer, a value of $ck/H = 0.195$ is proposed. It was shown that this value is approximately valid for streams wider than $D/H = 0.5$; small errors on the order of 2% are introduced, but the errors are nearly independent of the flow conditions, making both model calibration and predictions meaningful. For cases where the stream is very narrow or where no error in the mathematical model is acceptable, we presented an alternate Dupuit model of groundwater–surface water interaction that is capable of eliminating all global error in heads and discharges. Two parameters are required to accomplish this: the resistance of the stream bed and the hydraulic conductivity beneath the stream were chosen. We have shown that the effective values for this specific combination of parameters are entirely independent of the regional flow conditions and only weakly dependent on the problem geometry. In addition, we have provided explicit equations to evaluate the two effective parameters. Finally, we investigated the local behavior of the Dupuit flow field beneath the stream. We found by comparing the exact and Dupuit transition diagrams that the Dupuit solution provides a reasonable representation of the exact, local conditions.

We have found that Dupuit models can be very accurate when applied with insight to problems containing regions of concentrated vertical flow. The approach presented here is particularly useful when the details of the groundwater–stream interaction are not the focus of the model, but when streams occur throughout the model as boundary conditions. We may model the streams accurately and in a simple fashion without introducing global errors into the model, and without the computational burden associated with vertical discretization. In practice, standard numerical packages for solving problems of two-dimensional groundwater flow often include the capability to model heterogeneities and leaky beds; to apply the results presented here the effective properties, $c$ and $k'$, can be evaluated from the geometric ratio $D/H$ and assigned directly to the two-dimensional model without having to modify any computer code.

The problem considered here is two-dimensional, steady, and confined. However, the features of the effective values that include a weak dependence on geometry and no dependence on the distribution of regional flow suggest that the results are applicable, at least approximately, in more general settings; for example, settings where the stream alignment is not straight, where the stream width varies along the length of the stream, and where the hydraulic head in the stream decreases in the direction of stream flow. Numerical work is necessary to verify this. In many instances, a real leaky stream bed consisting of low permeability sediments or silts lines the stream. In these cases we anticipate that the accuracy of the Dupuit model improves as the head
losses due to vertical flow in the aquifer are small when compared to the losses due to the leaky stream bed.

Appendix A

We demonstrate that the algebraic equations (39) and (40) may be solved explicitly for the effective parameters. First, we separate the term \((\lambda / H)(k/k^*)\pi\) from each equation (39) and (40), to obtain

\[
\frac{\lambda}{H} \frac{k}{k^*} \pi = \frac{\ln \left[ \frac{1}{4} (1 - \gamma) \right] - \frac{k}{k^*} \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] \sinh \left( \frac{D}{\lambda} \right)}{\frac{k}{k^*} - \cosh \left( \frac{D}{\lambda} \right)} 
\]

(52)

\[
\frac{\lambda}{H} \frac{k}{k^*} \pi = \frac{\ln \left[ \frac{1}{4} (1 - \gamma) \right] - \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right] \sinh \left( \frac{D}{\lambda} \right)}{1 - \frac{k}{k^*} \cosh \left( \frac{D}{\lambda} \right)} 
\]

(53)

Subtracting (53) from (52), clearing the denominators, and dividing by \(\sinh(D/\lambda)\), we obtain

\[
\ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \left( \frac{Q_l}{Q_h} \right)^2} \right] = \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \cosh \left( \frac{D}{\lambda} \right) \left[ 1 - \left( \frac{Q_l}{Q_h} \right)^2 \right] \right] 
\]

(54)

We now eliminate the common factor containing the parameter \(Q_l/Q_h\) to obtain

\[
cosh \left( \frac{D}{\lambda} \right) = \frac{\ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \right]}{\ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \right]} 
\]

(55)

which can be solved directly to obtain (41). To evaluate the second effective parameter, \(k/k^*\), we first solve both (39) and (40) for \(Q_l/Q_h\) to obtain

\[
\frac{Q_l}{Q_h} = \frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) + \ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right] 
\]

(56)

\[
\frac{Q_l}{Q_h} = \frac{\frac{1}{4} (1 - \gamma)}{\sinh \left( \frac{D}{\lambda} \right)} \sinh \left( \frac{D}{\lambda} \right) + \frac{\lambda}{H} \frac{k}{k^*} \pi \left[ \cosh \left( \frac{D}{\lambda} \right) - 1 \right] \right] 
\]

(57)

Equating (56) and (57) yields

\[
\ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right] + \ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \cosh \left( \frac{D}{\lambda} \right) \left[ 1 - \left( \frac{Q_l}{Q_h} \right)^2 \right] \right] 
\]

\[
= - \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right] + \left[ \frac{\lambda}{H} \frac{k}{k^*} \pi \right]^2 \cosh \left( \frac{D}{\lambda} \right) \left( \cosh \left( \frac{D}{\lambda} \right) - 1 \right] 
\]

\[
\times \left\{ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right\} 
\]

(58)

At this point, it is easier to verify the solution, (41) and (42), than to derive the results. We use the identity (55) to verify the solution (42), by substituting both into (58). We consider the right-hand side of (58) first, and make the following substitution obtained from (42)

\[
\frac{\lambda}{H} \frac{k}{k^*} \pi = - \ln \left[ \frac{1 - \gamma^{1/2}}{1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right] 
\]

(59)

Substitution into the right-hand side of (58) yields zero; all terms on the right-hand side cancel. For the solution to be correct, the left hand side must vanish also. From (55) and (59) we obtain

\[
\ln \left[ \frac{\frac{1}{4} (1 - \gamma)}{1 - \gamma^{1/2} / 1 + \gamma^{1/2}} \sinh \left( \frac{D}{\lambda} \right) \right] = - \frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) 
\]

(60)

Substitution of (60) into the left-hand side of (58) yields

\[
- \frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) \left\{ 2 \frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) \right\} 
\]

\[
- \frac{\lambda}{H} \frac{k}{k^*} \pi \cosh \left( \frac{D}{\lambda} \right) - \frac{\lambda}{H} \frac{k}{k^*} \pi \right\} 
\]

\[
+ \frac{\frac{\lambda}{H} \frac{k}{k^*} \pi}{\cosh \left( \frac{D}{\lambda} \right) \cosh \left( \frac{D}{\lambda} \right) - 1 \right] = 0 
\]

(61)

The right-hand side reduces to zero as well, demonstrating that the solution, (41) and (42), is correct.

References


