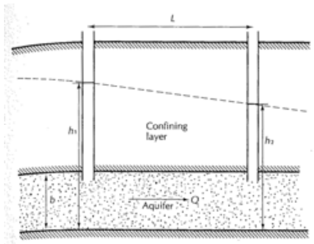
Confined Aquifer Steady Flow Calculations



◄ FIGURE 4.16
Steady flow through a confined aquifer of uniform thickness.

Confined Aquifer Steady Flow Calculations

The quantity of flow per unit width, q', may be determined from Darcy's law:

$$q' = Kb\frac{dh}{dl} \qquad (4.55)$$

where

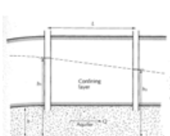
q' is the flow per unit width (L^2/T; ft^2/d or m^2/d)
K is the hydraulic conductivity (L/T; ft/d or m/d)
b is the aquifer thickness (L; ft or m)
$\frac{dh}{dl}$ is the slope of potentiometric surface (dimensionless)

One may wish to know the head, h (L; ft or m), at some intermediate distance, x (L; ft or m), between h_1 and h_2. This may be found from the equation

$$h = h_1 - \frac{q'}{Kb}x \qquad (4.56)$$

where x is the distance from h_1.

Confined Aquifer Steady Flow Calculations



◄ FIGURE 4.16
Steady flow through a confined aquifer of uniform thickness.

A confined aquifer is 33 m thick and 7 km wide. Two observation wells are located 1.2 km apart in the direction of flow. The head in well 1 is 97.5 m and in well 2 it is 89.0 m. The hydraulic conductivity is 1.2 m/d. What is the total daily flow of water through the aquifer?

$$Q = -Kb\frac{dh}{dl} \times \text{width}$$

$$= 1.2\text{ m/day} \times 33\text{ m} \times \frac{97.5\text{ m} - 89.0\text{ m}}{1200\text{ m}} \times 7000\text{ m}$$

$$= 2000\text{ m}^3/\text{day} \qquad \text{(to two significant digits)}$$

What is the elevation of the potentiometric surface at a point located 0.3 km from well h_1 and 0.9 km from well h_2? Discharge per unit width is $(2000\text{ m}^3/\text{d})/(7000\text{ m}) = 0.29\text{ m}^2/\text{day}$:

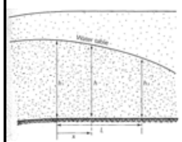
$$h = h_1 - \frac{q'}{Kb}x$$

$$= 97.5\text{ m} - \frac{0.29\text{ m}^2/\text{day}}{1.2\text{ m/d} \times 33\text{ m}} \times 300\text{ m}$$

$$= 97.5\text{ m} - 2.2\text{ m}$$

$$= 95.3\text{ m}$$

Unconfined Aquifer Steady Flow Calculations



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

This problem was solved by Dupuit (1863), and his assumptions are known as the **Dupuit assumptions.** The assumptions are that (1) the hydraulic gradient is equal to the slope of the water table and (2) for small water-table gradients, the streamlines are horizontal and the equipotential lines are vertical. Solutions based on these assumptions have proved to be useful in many practical problems. However, the Dupuit assumptions do not allow for a seepage face above the outflow side.

From Darcy's law,

$$q' = -Kh\frac{dh}{dx} \quad (4.57)$$

where h is the saturated thickness of the aquifer. At $x = 0$, $h = h_1$; at $x = L$, $h = h_2$.

Equation 4.57 may be set up for integration with the boundary conditions:

$$\int_0^L q'dx = -K\int_{h_1}^{h_2} hdh$$

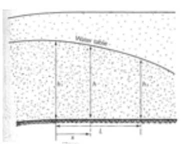
Integration of the preceding yields

$$q'x\Big|_0^L = -K\frac{h^2}{2}\Big|_{h_1}^{h_2}$$

Substitution of the boundary conditions for x and h yields

$$q'L = -K\left(\frac{h_2^2}{2} - \frac{h_1^2}{2}\right) \quad (4.58)$$

Unconfined Aquifer Steady Flow Calculations



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

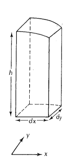
Rearrangement of Equation 4.58 yields the **Dupuit equation:**

$$\boxed{q' = \frac{1}{2}K\left(\frac{h_1^2 - h_2^2}{L}\right)} \quad (4.59)$$

where

- q' is the flow per unit width (L^2/T; ft^2/d or m^2/d)
- K is the hydraulic conductivity (L/T; ft/d or m/d)
- h_1 is the head at the origin (L; ft or m)
- h_2 is the head at L (L; ft or m)
- L is the flow length (L; ft or m)

▲ FIGURE 4.18
Control volume for flow through a prism of an unconfined aquifer with the bottom resting on a horizontal impervious surface and the top coinciding with the water table.



If we consider a small prism of the unconfined aquifer, it will have the shape of Figure 4.18. On one side it is h units high and slopes in the x-direction. Given the Dupuit assumptions, there is no flow in the z-direction. The flow in the x-direction, per unit width, is q'_x. From Darcy's law, the total flow in the x-direction through the left face of the prism is

$$q'_x dy = -K\left(h\frac{\partial h}{\partial x}\right)_x dy \quad (4.60)$$

where dy is the width of the face of the prism. The discharge through the right face, q'_{x+dx} is

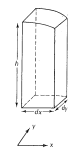
$$q'_{x+dx}dy = -K\left(h\frac{\partial h}{\partial x}\right)_{x+dx} dy \quad (4.61)$$

Note that $\left(h\frac{\partial h}{\partial x}\right)$ has different values at each face. The change in flow rate in the x-direction between the two faces is given by

$$(q'_{x+dx} - q'_x)dy = -K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dx\,dy \quad (4.62)$$

Through a similar process, it can be shown that the change in the flow rate in the y-direction is

$$(q'_{y+dy} - q'_y)dx = -K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dy\,dx \quad (4.63)$$



▲ FIGURE 4.18
Control volume for flow through a prism of an unconfined aquifer with the bottom resting on a horizontal impervious surface and the top coinciding with the water table.

For steady flow, any change in flow through the prism must be equal to a gain or loss of water across the water table. This could be infiltration or evapotranspiration. The net addition or loss is at a rate of w, and the volume change within the initial volume is $w\,dx\,dy$ where $dx\,dy$ is the area of the surface. If w represents evapotranspiration, it will have a negative value. As the change in flow is equal to the new addition,

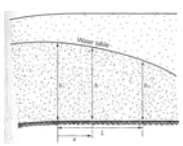
$$-K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dx\,dy - K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dy\,dx = w\,dx\,dy \qquad \textbf{(4.64)}$$

We can simplify Equation 4.64 by dropping out $dx\,dy$ and combining the differentials:

$$-K\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2}\right) = 2w \qquad \textbf{(4.65)}$$

If $w = 0$, then Equation 4.65 reduces to a form of the Laplace equation:

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0 \qquad \textbf{(4.66)}$$



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

If flow is in only one direction and we align the x-axis parallel to the flow, then there is no flow in the y-direction, and Equation 4.65 becomes

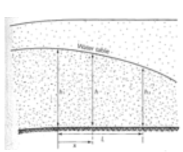
$$\frac{d^2(h^2)}{dx^2} = -\frac{2w}{K} \qquad \textbf{(4.67)}$$

Integration of this equation yields the expression

$$h^2 = -\frac{wx^2}{K} + c_1x + c_2 \qquad \textbf{(4.68)}$$

where c_1 and c_2 are constants of integration.

The following boundary conditions can be applied: at $x = 0$, $h = h_1$; at $x = L$, $h = h_2$ (Figure 4.19). By substituting these into Equation 4.68, the constants of integration can be evaluated with the following result:



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

$$h^2 = h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L - x)x \qquad \textbf{(4.69)}$$

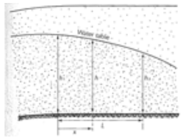
or

$$\boxed{h = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L - x)x}} \qquad \textbf{(4.70)}$$

where

- h is the head at x (L; ft or m)
- x is the distance from the origin (L; ft or m)
- h_1 is the head at the origin (L; ft or m)
- h_2 is the head at L (L; ft or m)
- L is the distance from the origin at the point h_2 is measured (L; ft or m)
- K is the hydraulic conductivity (L/T; ft/d or m/d)
- w is the recharge rate (L/T; ft/d or m/d)

This equation can be used to find the elevation of the water table anywhere between two points located L distance apart if the saturated thickness of the aquifer is known at the two end points.



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

For the case in which there is no infiltration or evaporation, $w = 0$ and Equation 4.70 reduces to

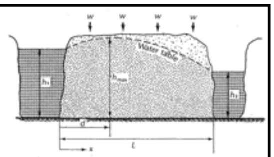
$$h = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L}} \tag{4.71}$$

By differentiating Equation 4.69, and because $q'_x = -Kh(dh/dx)$, it may be shown that the discharge per unit width, q'_x, at any section x distance from the origin is given by

$$q'_x = \frac{K(h_1^2 - h_2^2)}{2L} - w\left(\frac{L}{2} - x\right) \tag{4.72}$$

where

- q'_x is the flow per unit width at x (L^2/T; ft²/d or m²/d)
- x is the distance from the origin (L; ft or m)
- K is the hydraulic conductivity (L/T; ft/d or m/d)
- h_1 is the head at the origin (L; ft or m)
- h_2 is the head at L (L; ft or m)
- L is the distance from the origin at the point where h_2 is measured (L; ft or m)
- w is the recharge rate (L/T; ft/d or m/d)



If the water table is subject to infiltration, there may be a water divide with a crest in the water table. In this case, q'_x will be zero at the water divide. If d is the distance from the tion. origin to a water divide, then substituting $q'_x = 0$ and $x = d$ into Equation 4.72 yields

$$d = \frac{L}{2} - \frac{K}{w}\frac{(h_1^2 - h_2^2)}{2L} \tag{4.73}$$

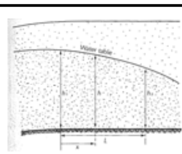
where

- d is the distance from origin to water divide (L; ft or m)
- h_1 is the head at the origin (L; ft or m)
- h_2 is the head at L (L; ft or m)
- L is the distance from the origin where h_2 is measured (L; ft or m)
- K is the hydraulic conductivity (L/T; ft/d or m/d)
- w is the recharge rate (L/T; ft/d or m/d)

Once the distance from the origin to the water divide has been found, then the elevation of the water table at the divide may be determined by substituting d for x in Equation 4.70.

$$h_{max} = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)d}{L} + \frac{w}{K}(L - d)d} \tag{4.74}$$

Where h_{max} is the maximum elevation, which is the elevation at the water table divide.



◄ FIGURE 4.17
Steady flow through an unconfined aquifer resting on a horizontal impervious surface.

An unconfined aquifer has a hydraulic conductivity of 0.0020 cm/s and an effective porosity of 0.27. The aquifer is in a bed of sand with a uniform thickness of 31 m, as measured from the land surface. At well 1, the water table is 21 m below the land surface. At well 2, located some 175 m away, the water table is 23.5 m from the surface. What are (A) the discharge per unit width, (B) the average linear velocity at well 1, and (C) the water-table elevation midway between the two wells?

Part A: From Equation 4.59,

$$q' = K\frac{(h_1^2 - h_2^2)}{2L}$$

$h_1 = 31\text{ m} - 21\text{ m} = 10\text{ m}$

$h_2 = 31\text{ m} - 23.5\text{ m} = 7.5\text{ m}$

$L = 175\text{ m}$

$$q' = 1.7\text{ m/d} \times \frac{10^2\text{ m}^2 - 7.5^2\text{ m}^2}{2 \times 175\text{ m}}$$

$= 0.21\text{ m}^2\text{/d}$ per unit width

Part B: From Equation 4.24,

$$v_x = \frac{Q}{n_e A}$$

As $Q = q' \times$ unit width and $A = h_1 \times$ unit width,

$$v_x = \frac{q'}{n_e h_1}$$

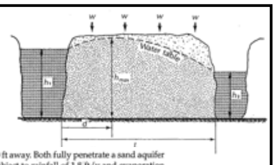
$$= \frac{0.21\text{ m}^2\text{/d}}{0.27 \times 10\text{ m}} = 0.078\text{ m/day}$$

Part C: From Equation 4.71,

$$h = \sqrt{h_1^2 - (h_1^2 - h_2^2)\frac{x}{L}}$$

$$= \sqrt{(10\text{ m})^2 - [(10\text{ m})^2 - (7.5\text{ m})^2]\left(\frac{87.5\text{ m}}{175\text{ m}}\right)}$$

$= 8.8\text{ m}$



A canal was constructed running parallel to a river 1500 ft away. Both fully penetrate a sand aquifer with a hydraulic conductivity of 1.2 ft/d. The area is subject to rainfall of 1.8 ft/y and evaporation of 1.3 ft/y. The elevation of the water above the base of the aquifer in the river is 31 ft and in the canal it is 27 ft. Determine (A) the water divide, (B) the maximum water-table elevation, (C) the daily discharge per 1000 ft into the river, and (D) the daily discharge per 1000 ft into the canal. tion.

Part A: From Equation 4.73,

$$d = \frac{L}{2} - \frac{K}{w}\frac{(h_1^2 - h_2^2)}{2L}$$

$h_1 = 31$ ft

$h_2 = 27$ ft

$L = 1500$ ft

$K = 1.2$ ft/d

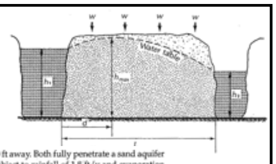
$w = 1.8$ ft/y infiltration $-$ 1.3 ft/y evaporation

$= 0.50$ ft/y accretion

$= 0.0014$ ft/day

$$d = \frac{1500\text{ ft}}{2} - \frac{1.2\text{ ft/d}}{0.0014\text{ ft/d}}\left(\frac{(31\text{ ft})^2 - (27\text{ ft})^2}{2 \times 1500\text{ ft}}\right)$$

$= 680$ ft from the river



A canal was constructed running parallel to a river 1500 ft away. Both fully penetrate a sand aquifer with a hydraulic conductivity of 1.2 ft/d. The area is subject to rainfall of 1.8 ft/y and evaporation of 1.3 ft/y. The elevation of the water above the base of the aquifer in the river is 31 ft and in the canal it is 27 ft. Determine (A) the water divide, (B) the maximum water-table elevation, (C) the daily discharge per 1000 ft into the river, and (D) the daily discharge per 1000 ft into the canal. tion.

Part B: From Equation 4.74,

$$h_{max} = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)d}{L} + \frac{w}{K}(L - d)d}$$

$$= \sqrt{(31\text{ ft})^2 - \frac{[(31\text{ ft})^2 - (27\text{ ft})^2]\,680\text{ ft}}{1500\text{ ft}} + \frac{0.0014\text{ ft/d}}{1.2\text{ ft/d}}(1500\text{ ft} - 680\text{ ft})\,680\text{ ft}}$$

$= 39$ ft

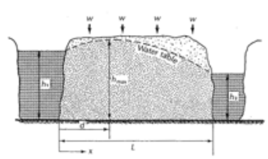
Part C: From Equation 4.72, for $x = 0$:

$$q_x = \left[\frac{K(h_1^2 - h_2^2)}{2L} - w\left(\frac{L}{2} - x\right)\right] \times \text{width}$$

$$= \left[\frac{(1.2\text{ ft/day})[(31\text{ ft})^2 - (27\text{ ft})^2]}{2 \times 1500\text{ ft}} - (0.0014\text{ ft/d})\left(\frac{1500\text{ ft}}{2} - 0\right)\right] \times 1000\text{ ft}$$

$= -960\text{ ft}^3/\text{d}$

The negative sign indicates that flow is in the opposite direction of x, or into the river.



A canal was constructed running parallel to a river 1500 ft away. Both fully penetrate a sand aquifer with a hydraulic conductivity of 1.2 ft/d. The area is subject to rainfall of 1.8 ft/y and evaporation of 1.3 ft/y. The elevation of the water above the base of the aquifer in the river is 31 ft and in the canal it is 27 ft. Determine (A) the water divide, (B) the maximum water-table elevation, (C) the daily discharge per 1000 ft into the river, and (D) the daily discharge per 1000 ft into the canal.

Part D: From Equation 4.72,

$x = L$

$$q_x = \left[\frac{K(h_1^2 - h_2^2)}{2L} - w\left(\frac{L}{2} - x\right)\right] \times \text{width}$$

$$= \left[\frac{(1.2\text{ ft/day})[(31\text{ ft})^2 - (27\text{ ft})^2]}{2 \times 1500\text{ ft}} - (0.0014\text{ ft/d})\left(\frac{1500\text{ ft}}{2} - 1500\text{ ft}\right)\right] \times 1000\text{ ft}$$

$= 1100\text{ ft}^3/\text{d}$

Flow is in the direction of x, or into the canal.