Mechanical Energy: Kinetic

4.2 Mechanical Energy

There are a number of different types of mechanical energy recognized in classical physics. Of these, we will consider kinetic energy, gravitational potential energy, and energy of fluid pressures.

A moving body or fluid tends to remain in motion, according to Newtonian physics, because it possesses energy due to its motion called *kinetic energy.* This energy is equal to one-half the product of its mass and the square of the magnitude of the velocity:

$$E_k = \tfrac{1}{2}mv^2 \qquad (4.1)$$

where

- E_k is the kinetic energy (ML^2/T^2; slug-ft^2/s^2 or kg·m^2/s^2)
- v is the velocity (L/T; ft/s or m/s)
- m is the mass (M; slug or kg)

If m is in kilograms and v is in meters per second, then E_k has the units of kg·m^2/s^{2d} or newton-meters. The unit of energy is the joule, which is one newton-meter. The joule is also the unit of work.

Mechanical Energy: Gravity

Imagine that a weightless container filled with water of mass m is moved vertically upward a distance, z, from some reference surface (a datum). Work is done in moving the mass of water upward. This work is equal to

$$W = Fz = (mg)z \qquad (4.2)$$

where

- W is work (ML^2/T^2; slug-ft^2/s^2 or kg·m^2/s^2)
- z is the elevation of the center of gravity of the fluid above the reference elevation (L; ft or m)
- m is the mass (M; slugs or kg)
- g is the acceleration of gravity (L/T^2; ft/s^2 or m/s^2)
- F is a force (ML/T^2; slug-ft/s^2 or kg·m/s^2)

The mass of water has now acquired energy equal to the work done in lifting the mass. This is a potential energy, due to the position of the fluid mass with respect to the datum. E_g is *gravitational potential energy:*

$$W = E_g = mgz \qquad (4.3)$$

Mechanical Energy: Pressure

A fluid mass has another source of potential energy owing to the **pressure** of the surrounding fluid acting upon it. Pressure is the force per unit area acting on a body:

$$P = F/A \qquad (4.4)$$

where

- P is the pressure [M/LT^2; slug-ft/s^2 or (kg·m/s^2)/m^2]
- A is the cross-sectional area perpendicular to the direction of the force (L^2; ft^2 or m^2)

The units of pressure are pascals (Pa), or N/m^2. A N/m^2 is equal to a N·m/m^3, or J/m^3. Pressure may thus be thought of as potential energy per unit volume of fluid.

For a unit volume of fluid, the mass, m, is numerically equal to the density, ρ, since density is defined as mass per unit volume. The total energy of the unit volume of fluid is the sum of the three components—kinetic, gravitational, and fluid-pressure energy:

$$E_{tv} = \tfrac{1}{2}\rho v^2 + \rho gz + P \qquad (4.5)$$

Bernoulli Equation

If Equation 4.5 is divided by ρ, the result is total energy per unit mass, E_{tm}:

$$E_{tm} = \frac{v^2}{2} + gz + \frac{P}{\rho} \quad (4.6)$$

which is known as the *Bernoulli equation*. The derivation of the Bernoulli equation may be found in textbooks on fluid mechanics (Hornberger, Raffensperger, Wilberg, and Eshleman, 1998).

For steady flow of a frictionless, incompressible fluid along a smooth line of flow, the sum of the three components is a constant. Each term of Equation 4.6 has the units of $(L/T)^2$:

$$\frac{v^2}{2} + gz + \frac{P}{\rho} = \text{constant} \quad (4.7)$$

Steady flow indicates that the conditions do not change with time. The density of an incompressible fluid would not change with changes in pressure. A frictionless fluid would not require energy to overcome resistance to flow. An ideal fluid would have both of these characteristics; real fluids have neither one. Real fluids are compressible and do suffer frictional flow losses; however, Equation 4.7 is useful for purposes of comparing the components of mechanical energy.

If each term of Equation 4.7 is divided by g, the following expression results:

$$\frac{v^2}{2g} + z + \frac{P}{\rho g} = \text{constant} \quad (4.8)$$

Example

PROBLEM

At a place where $g = 9.80\ \text{m/s}^2$ the fluid pressure is $1500\ \text{N/m}^2$; the distance above a reference elevation is 0.75 m; and the fluid density is $1.02 \times 10^3\ \text{kg/m}^3$. The fluid is moving at a velocity of 1.0×10^{-6} m/s. Find E_{tm}.

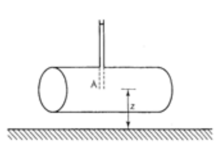
$$\begin{aligned} E_{tm} &= gz + \frac{P}{\rho} + \frac{v^2}{2} \\ &= 9.80\ \text{m/s}^2 \times 0.75\ \text{m} + \frac{1500\ \text{N/m}^2}{1.02 \times 10^3\ \text{kg/m}^3} + \frac{(10^{-6})^2}{2}\frac{\text{m}^2}{\text{s}^2} \\ &= 7.35\ \text{m}^2/\text{s}^2 + 1.47\ \text{m}^2/\text{s}^2 + 5.0 \times 10^{-13}\ \text{m}^2/\text{s}^2 \\ &= 8.8\ \text{m}^2/\text{s}^2 \end{aligned}$$

Kinetic Energy is small!

The total energy per unit mass of $8.8\ \text{m}^2/\text{s}^2$ is almost exclusively in the pressure and gravitational potential energy terms, which are 13 orders of magnitude greater than the value of kinetic energy.

Hydraulic Head

► FIGURE 4.1
Piezometer measuring fluid pressure and the elevation of water.



The preceding problem shows that the amount of energy developed as kinetic energy by flowing ground water is small. The velocity of ground water flowing in porous media under natural hydraulic gradients is very low. The example velocity of 10^{-6} m/s results in a movement of 30 m/y, which is typical for ground water.

Velocity components of energy may be safely ignored in ground-water flow because they are so much smaller than the other two terms. By dropping $v^2/2g$ from Equation 4.8, the total hydraulic head, h, is given by the formula

$$h = z + \frac{P}{\rho g} \quad (4.9)$$

Hydraulic Head

► FIGURE 4.2
Total head, h, elevation head, z, and pressure head, h_p.

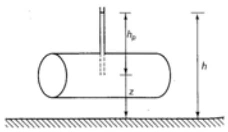


Figure 4.2 shows the components of head. The head is the total mechanical energy per unit weight of water. For a fluid at rest, the pressure at a point is equal to the weight of the overlying water per unit cross-sectional area:

$$P = \rho g h_p \quad \textbf{(4.10)}$$

where h_p is the height of the water column that provides a *pressure head*. Substituting into Equation 4.9, we see that

$$h = z + h_p \quad \textbf{(4.11)}$$

The total hydraulic head is equal to the sum of the elevation head and the pressure head. The elevation and pressure heads, when used in the form of Equation 4.11, correlate with energy per unit weight of water with dimensions L.

Hydraulic Head – Example 1

Two points in the same confined aquifer are located on a vertical line. Point 1 is 100 m below mean sea level and point 2 is 50 m below mean sea level. The fluid pressure at point 1 is 9.0×10^5 N/m² and at point 2, it is 6.1×10^5 N/m².

Part A: Calculate the pressure and total heads at each point.

Assume that the deeper point is at zero datum. Therefore, the elevation head at point 1 is zero and at point 2 is 50 m.

By rearranging Equation 4.10 we can get an equation for the pressure head:

$$P = \rho g h_p$$

Therefore,

$$h_p = P/\rho g$$

Assume that $g = 9.80$ m/s² and $\rho = 1000$ kg/m³. At point 1,

$$h_p = \frac{9.0 \times 10^5\ (\text{kg·m/s}^2)/(\text{m}^2)}{1000\ \text{kg/m}^3 \times 9.80\ \text{m/s}^2}$$
$$= 92\ \text{m}$$

Since total head is the sum of the elevation head and the pressure head, at point 1

$$h = h_p + z = 92\ \text{m} + 0\ \text{m} = 92\ \text{m}$$

At point 2,

$$h_p = \frac{6.1 \times 10^5 (\text{kg·m/s}^2)/(\text{m}^2)}{1000\ \text{kg/m}^3 \times 9.80\ \text{m/s}^2}$$
$$= 62\ \text{m}$$
$$h = 62\ \text{m} + 50\ \text{m} = 112\ \text{m}$$

Part B: Does flow in the aquifer have an upward or downward component?

Flow is downward, because the total head at 50 m below sea level is greater than the total head at 100 m below sea level, even though the pressure head at 100 m is greater.

Hydraulic Head – Example 2

The following data were collected at a nest of piezometers [several piezometers of different depths located within a few feet (1 to 2 m) of each other]:

	A	B	C
Elevation at surface (m a.s.1.)	225	225	225
Depth of piezometer (m)	150	100	75
Depth to water (m below surface)	80	77	60

Part A: What is the hydraulic head at A, B, and C?

Hydraulic head is elevation of the water in the piezometer. It is found by subtracting the depth to water from the surface elevation.

A: 145 m B: 148 m C: 165 m

Hydraulic Head – Example 2

Part B: What is the pressure head at A, B, and C?

Pressure head is the height of the water in the well above the depth of the piezometer. It is found by subtracting the depth to water from the depth of the piezometer from the surface.

A: 70 m B: 23 m C: 15 m

Part C: What is the elevation head in each well?

Elevation head is the height of the measuring point above the datum. In this case the datum is mean sea level and the elevation head is found by subtracting the depth of the piezometer from the surface elevation.

A: 75 m B: 125 m C: 150 m

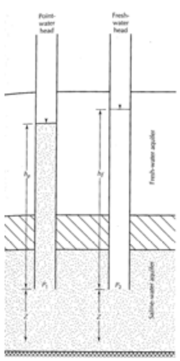
Notice that the total head found in part A is the sum of the pressure head found in part B and the elevation head found in part C.

Part D: What is the vertical hydraulic gradient between the piezometers?

The hydraulic gradient is the difference in total head divided by the vertical distance between the two piezometers.

From piezometer A to piezometer B the difference in the total head is 148 m–145 m and the vertical distance is 50 m. The hydraulic gradient is (3 m)/(50 m), or 0.06, and the direction is downward as the head in B, the shallower piezometer, is greater.

From piezometer B to piezometer C the difference in total head is 165 m–148 m and the vertical distance is 25 m. The hydraulic gradient is (17 m)/(25 m), or 0.68. This gradient is also downward.



◄ FIGURE 4.3
Definition of point-water head and fresh-water head.

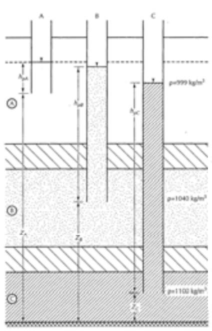
Consider piezometer 1 of Figure 4.3, which is filled with point water of density ρ_p. The point-water pressure head is the height of the point water in the piezometer, h_p. The pressure at P_1 may be found from

$$P_1 = \rho_p g h_p$$

$$P_2 = \rho_f g h_f$$

$$\rho_p g h_p = \rho_f g h_f$$

$$h_f = (\rho_p/\rho_f) h_p$$



▲ FIGURE 4.4
Point-water heads for a system of three aquifers, each containing water with a different density.

Figure 4.4 shows three aquifers—a fresh-water-table aquifer and two confined saline-wat[er] aquifers, with properties as given in the following table. What are the head relationships betwe[en] aquifers?

Aquifer	Water Density	Elevation Head	Point-Water Head
A	999 kg/m³	50.00 m	55.00 m
B	1040 kg/m³	31.34 m	54.67 m
C	1100 kg/m³	7.95 m	51.88 m

Field measurements of water levels in the wells give the point-water heads. The point-water pressure heads are determined by subtracting the elevation head from the point-water head. The fresh-water pressure head is then found by multiplying the point-water pressure head by the ratio of the point-water density to the fresh-water density. Finally, the fresh-water pressure head is added to the elevation head to obtain fresh-water heads as given in the following table.

Aquifer	Point-Water Pressure Head	ρ_p/ρ_f	Fresh-Water Pressure Head	Fresh-Water Head
A	5.00 m	1.00	5.00 m	55.00 m
B	23.33 m	1.04	24.3 m	55.5 m
C	43.93 m	1.10	48.3 m	56.3 m

Note that the point-water heads indicate that the hydraulic gradient is vertically downward. However, after the point-water heads are corrected to fresh-water heads, the true hydraulic gradient is seen to be upward.

4.5 Force Potential and Hydraulic Head

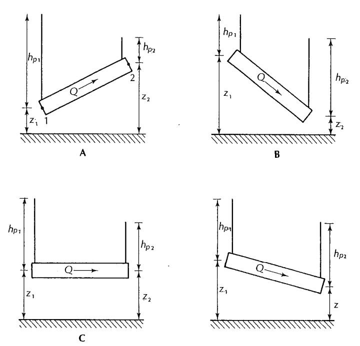
In Equation 4.6 we showed the total mechanical energy per unit mass to be equal to the sum of the kinetic energy, elevation energy, and pressure. This total potential energy has been termed the **force potential** and is indicated by the capital Greek letter phi, Φ (Hubbert 1940):

$$\Phi = gz + \frac{P}{\rho} = gz + \frac{\rho g h_p}{\rho} = g(z + h_p) \quad \textbf{(4.16)}$$

Since $z + h_p = h$, the hydraulic head.

$$\Phi = gh \quad \textbf{(4.17)}$$

Fluid Flows from High Head to Low Head



▲ FIGURE 4.5
Apparatus to demonstrate how changing the slope of a pipe packed with sand will change the components of elevation, z, and pressure, h_p, heads. The direction of flow, Q, is indicated by the arrow.

4.6.1 Darcy's Law in Terms of Head and Potential

In Section 3.5 it was shown that flow through a pipe filled with sand is proportional to the decrease in hydraulic head divided by the length of the pipe. This ratio is called the *hydraulic gradient*. It should now be apparent that the hydraulic head is the sum of the pressure head and the elevation head. Expressed in terms of hydraulic head, Darcy's law is

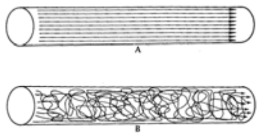
$$Q = -KA\frac{dh}{dl} \quad \textbf{(4.18)}$$

Since the fluid potential, Φ, is equal to gh, Darcy's law can also be expressed in terms of potential as (Hubbert 1940)

$$Q = -\frac{KA}{g}\frac{d\Phi}{dl} \quad \textbf{(4.19)}$$

As expressed here, Darcy's law is in a one-dimensional form, as water flows through the pipe in only one direction. In later sections, we will examine various forms of Darcy's law for two and three directions.

Validity of Darcy’s Law



▲ FIGURE 4.6
A. Flow paths of molecules of water in laminar flow. **B.** Flow paths of molecules of water in turbulent flow.

The **Reynolds number** relates the four factors that determine whether the flow will be laminar or turbulent (Hornberger, Raffensperger, Wilberg, and Eshleman, 1998):

$$R = \frac{\rho q d}{\mu} \qquad (4.20)$$

where

- R is the Reynolds number, dimensionless
- ρ is the fluid density (M/L^3; kg/m^3)
- q is the discharge velocity (L/T; m/s)
- d is the diameter of the passageway through which the fluid moves (L; m)
- μ is the viscosity ($M/T{\cdot}L$; kg/s·m)

Validity of Darcy’s Law

A sand aquifer has a median grain diameter of 0.050 cm. For pure water at 15°C, what is the greatest velocity for which Darcy’s law is valid?

From Appendix 14,

$$\rho = 0.999 \times 10^3 \text{ kg/m}^3$$
$$\mu = 1.14 \times 10^{-2} \text{ g/s·cm}$$

Convert units to kilograms, meters, and seconds:

$$d = 0.050 \text{ cm} \times 0.01 \text{ m/cm} = 0.0005 \text{ m}$$
$$\mu = 1.14 \times 10^{-2} \text{ g/s·cm} \times 0.001 \text{ kg/g} \times 100 \text{ cm/m}$$
$$= 1.14 \times 10^{-3} \text{ kg/s·m}$$

By rearranging Equation 4.16, we can obtain a formula for velocity:

$$R = \frac{\rho q d}{\mu}$$

Therefore,

$$q = \frac{R\mu}{\rho d}$$

If R cannot exceed 1, the maximum velocity is

$$q = \frac{1 \times 1.14 \times 10^{-3} \text{ kg/s·m}}{0.999 \times 10^3 \text{ kg/m}^3 \times 0.0005 \text{ m}}$$
$$= 0.0023 \text{ m/s}$$

Darcy’s law will be valid for discharge velocities equal to or less than 0.0023 m/s.

Specific Discharge and Average Linear Velocity

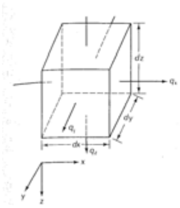
$$V = Q/A = -K(dh/dl)$$

$$V_x = (Q/n_e)A = -(K/n_e)(dh/dl)$$

where

- v_x is the average linear velocity (L/T; cm/s, ft/s, m/s)
- n_e is the effective porosity (dimensionless)

Equations of Groundwater Flow



◄ FIGURE 4.7
Control volume for flow through a confined aquifer.

We will consider a very small part of the aquifer, called a *control volume.* The three sides are of lengths dx, dy, and dz, respectively. The area of the faces normal to the x-axis is $dydz$; the area of the faces normal to the z-axis is $dxdy$ (Figure 4.7).

Assume the aquifer is homogeneous and isotropic. The fluid moves in only one direction through the control volume. However, the actual fluid motion can be subdivided on the basis of the components of flow parallel to the three principal axes. If q is flow per unit cross-sectional area, $\rho_w q_x$ is the portion parallel to the x-axis, etc., where ρ_w is the fluid density.

Equations of Groundwater Flow

The mass flux into the control volume is $\rho_w q_x\, dydz$ along the x-axis. The mass flux[†] out of the control volume is $\rho_w q_x\, dydz + \frac{\partial}{\partial x}(\rho_w q_x)\, dx\, dydz$. The net accumulation in the control volume due to movement parallel to the x-axis is equal to the inflow less the outflow, or $-\frac{\partial}{\partial x}(\rho_w q_x)\, dx\, dydz$. Since there are flow components along all three axes, similar terms can be determined for the other two directions: $-\frac{\partial}{\partial y}(\rho_w q_y)\, dy\, dxdz$ and $-\frac{\partial}{\partial z}(\rho_w q_z)\, dz\, dxdy$. Combining these three terms yields the net total accumulation of mass in the control volume:

$$-\left(\frac{\partial}{\partial x}\rho_w q_x + \frac{\partial}{\partial y}\rho_w q_y + \frac{\partial}{\partial z}\rho_w q_z\right)dxdydz \qquad \textbf{(4.25)}$$

The volume of water in the control volume is equal to $n\, dx\, dydz$, where n is the porosity. The initial mass of the water is thus $\rho_w n\, dx\, dydz$. The volume of solid material is $(1 - n)\, dx\, dydz$. Any change in the mass of water, M, with respect to time (t) is given by

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}(\rho_w n\, dxdydz) \qquad \textbf{(4.26)}$$

Equations of Groundwater Flow

As the pressure in the control volume changes, the fluid density will change, as will the porosity of the aquifer. The compressibility of water, β, is defined as the rate of change in density with a change in pressure, P:

$$\beta dP = \frac{d\rho_w}{\rho_w} \qquad \textbf{(4.27)}$$

The aquifer also changes in volume with a change in pressure. We will assume the only change is vertical. The aquifer compressibility, α, is given by

$$\alpha dP = \frac{d(dz)}{dz} \qquad \textbf{(4.28)}$$

As the aquifer compresses or expands, n will change, but the volume of solids, V_s, will be constant. Likewise, if the only deformation is in the z-direction, $d(dx)$ and $d(dy)$ will equal zero:

$$dV_s = 0 = d[(1 - n)\, dx\, dydz] \qquad \textbf{(4.29)}$$

Equations of Groundwater Flow

d[(1-n)dz] = 0

Differentiation of Equation 4.29 yields

$$dz\,dn = (1 - n)d(dz) \tag{4.30}$$

and

$$dn = \frac{(1 - n)d(dz)}{dz} \tag{4.31}$$

The pressure, P, at a point in the aquifer is equal to $P_0 + \rho_w gh$, where P_0 is atmospheric pressure, a constant, and h is the height of a column of water above the point. Therefore, $dP = \rho_w g\,dh$, and Equations 4.27 and 4.28, become

$$d\rho_w = \rho_w \beta(\rho_w g\,dh) \tag{4.32}$$

and

$$d(dz) = dz\alpha(\rho_w g\,dh) \tag{4.33}$$

Equation 4.31 can be rearranged if $d(dz)$ is replaced by Equation 4.33.

$$dn = (1 - n)\alpha\rho_w g\,dh \tag{4.34}$$

Equations of Groundwater Flow

If dx and dy are constant, the equation for change of mass with time in the control volume, Equation 4.26, can be expressed as

$$\frac{\partial M}{\partial t} = \left[\rho_w n \frac{\partial(dz)}{\partial t} + \rho_w dz \frac{\partial n}{\partial t} + n\,dz \frac{\partial \rho_w}{\partial t}\right] dxdy \tag{4.35}$$

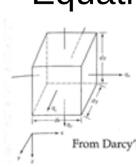
Substitution of Equations 4.32, 4.33, and 4.34 into Equation 4.35 yields, after minor manipulation,

$$\frac{\partial M}{\partial t} = (\alpha\rho_w g + n\beta\rho_w g)\rho_w\,dx\,dydz\,\frac{\partial h}{\partial t} \tag{4.36}$$

The net accumulation of material expressed as Equation 4.25 is equal to Equation 4.36, the change in mass with time:

$$-\left[\frac{\partial(q_x)}{\partial x} + \frac{\partial(q_y)}{\partial y} + \frac{\partial(q_z)}{\partial z}\right]\rho_w\,dx\,dydz = (\alpha\rho_w g + n\beta\rho_w g)\rho_w\,dx\,dydz\,\frac{\partial h}{\partial t} \tag{4.37}$$

Equations of Groundwater Flow



◄ FIGURE 4.7
Control volume for flow through a confined aquifer.

From Darcy's law,

$$q_x = -K\frac{\partial h}{\partial x} \tag{4.38}$$

$$q_y = -K\frac{\partial h}{\partial y} \tag{4.39}$$

and

$$q_z = -K\frac{\partial h}{\partial z} \tag{4.40}$$

Substituting these into Equation 4.37 yields the main equation of flow for a confined aquifer:

$$K\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = (\alpha\rho_w g + n\beta\rho_w g)\frac{\partial h}{\partial t} \tag{4.41}$$

Equations of Groundwater Flow

For two-dimensional flow with no vertical components, the equation can be rearranged and terms introduced from Equations 3–32 and 3–33 for the storativity, [$S = b(\alpha\rho_w g + n\beta\rho_w g)$], and from Equation 3.30 for the transmissivity, ($T = Kb$), where b is the aquifer thickness:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad (4.42)$$

In steady-state flow, there is no change in head with time, for example, in cases when there is no change in the position or slope of the water table. Under such conditions, time is not an independent variable, and steady flow is described by the three-dimensional partial differential equation known as the **Laplace equation:**

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \qquad (4.43)$$

Equations of Groundwater Flow: Leaky Aquifer

The leakage rate, or rate of accumulation, is designated as e. The general equation of flow (in two dimensions, since horizontal flow was assumed) is given by

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{e}{T} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad (4.44)$$

The leakage rate can be determined from Darcy's law. If the head at the top of the aquitard is h_0 and the head in the aquifer just below the aquitard is h, the aquitard has a thickness b' and a conductivity (vertical) of K':

$$e = K'\frac{(h_0 - h)}{b'} \qquad (4.45)$$

Equations of Groundwater Flow: Unconfined Aquifers

In the case of an unconfined aquifer, the saturated thickness can change with time. Under such conditions, the ability of the aquifer to transmit water—the transmissivity—changes, as it is the product of the conductivity, K, and the saturated thickness, h (assuming that h is measured from the horizontal base of the aquifer).

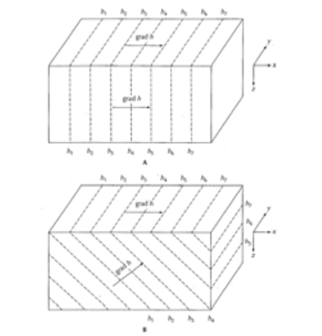
The general flow equation for two-dimensional unconfined flow is known as the **Boussinesq equation** (Boussinesq 1904):

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) = \frac{S_y}{K}\frac{\partial h}{\partial t} \qquad (4.46)$$

where S_y is specific yield. This equation is a type of differential equation that cannot be solved using calculus, except in some very specific cases. In mathematical terms, it is nonlinear.

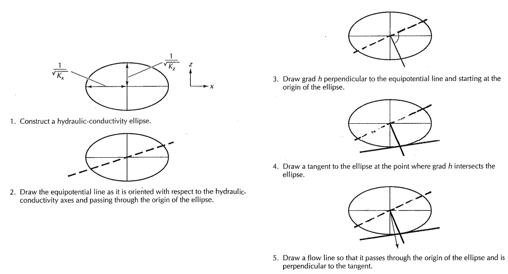
If the drawdown in the aquifer is very small compared with the saturated thickness, the variable thickness, h, can be replaced with an average thickness, b, that is assumed to be constant over the aquifer. The Boussinesq equation can thus be linearized by this approximation to the form

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S_y}{Kb}\frac{\partial h}{\partial t} \qquad (4.47)$$



▲ FIGURE 4.8
A. Grad *h* in a homogeneous, isotropic aquifer with a uniform horizontal flow field. **B.** Grad *h* in a homogeneous, isotopic aquifer with a uniform flow field with a downward component.

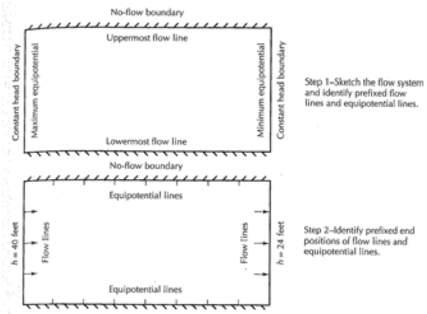
Anisotropic Media



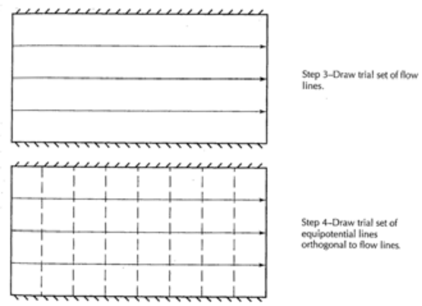
Anisotropic Media

▲ FIGURE 4.10
Relationship of flow lines to equipotential field and grad *h*. **A.** Isotropic aquifer. **B.** Anisotropic aquifer.

Flow Net Construction



Flow Net Construction



Rules for Flow Nets

1. Identify the boundary conditions.
2. Sketch the boundaries to scale with the two axes of the drawing having the same scale.
3. Identify the position of known equipotential and flow-line conditions.
4. Draw a trial set of flow lines. The outer flow lines will be parallel to no-flow boundaries. Flow nets do not need a finite boundary on all sides; it is possible to have a region of flow that extends beyond the outer edge of the flow net. A flow net can have a partial streamtube along one edge. The flow lines do not need to be spaced an equal distance apart.
5. Draw a trial set of equipotential lines. Start at one end of the flow field and work toward the other. The equipotential lines should be perpendicular to flow lines. They will be parallel to constant-head boundaries and at right angles to no-flow boundaries. If there is a water-table boundary, the position of the equipotential line at the water table is based on the elevation of the water table. The equipotential lines should be spaced so that they form four-sided shapes that have approximately equal central dimensions; that is, lines passing through the center of each shape should be of approximately the same length.
6. Erase and redraw the trial flow lines and equipotential lines until the desired flow net of orthogonal equipotential lines and flow lines is obtained.

Flow Net

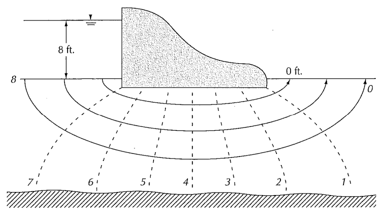


FIGURE 4.12 Flow net in an aquifer beneath an impervious dam.

Flow Net and q’

In addition to presenting a graphic display of the ground-water flow directions and potential distribution, the completed flow net can be used to determine the quantity of water flowing by the following formula:

$$q' = \frac{Kph}{f} \quad \textbf{(4.49)}$$

where

- q' is the total volume discharge per unit width of aquifer (L^3/T; ft^3/d or m^3/d)
- K is the hydraulic conductivity (L/T; ft/d or m/d)
- p is the number of flowtubes bounded by adjacent pairs of flow lines
- h is the total head loss over the length of the flow lines (L; ft or m)
- f is the number of squares bounded by any two adjacent flow lines and covering the entire length of flow.

Equation 4.49 can be used for simple flow systems with one recharge boundary and one discharge boundary. For complex systems, it is possible to find the discharge for each streamtube where $q' = (Kh)/f$. The total flow can be found by summing the flow in individual streamtubes.

Flow Net and q’

PROBLEM

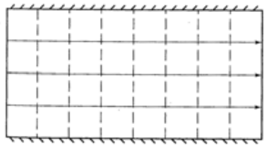
If hydraulic conductivity is 23 ft/day, what is the discharge per unit width of the flow system in Figure 4.11?

The number of streamtubes is 4; therefore, $p = 4$. The number of equipotential drops is 8; therefore, $f = 8$. The total head loss is 40 ft − 24 ft = 16 ft. Substituting these values into Equation 4.49:

$$q' = \frac{Kph}{f}$$

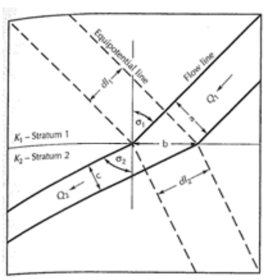
$$= \frac{23\ \text{ft/d} \times 4 \times 16\ \text{ft}}{8} \times 1\ \text{ft unit width}$$

$$= 180\ \text{ft}^3/\text{d}$$



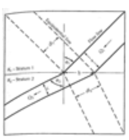
Step 4–Draw trial set of equipotential lines orthogonal to flow lines.

Streamtube



◄ FIGURE 4.13
Streamtube crossing a hydraulic conductivity boundary.

Streamtube



The flow through each streamtube is found from Darcy's law:

$$Q_1 = K_1 a \frac{dh_1}{dl_1} \quad \text{and} \quad Q_2 = K_2 c \frac{dh_2}{dl_2} \tag{4.50}$$

From the principle of continuity, Q_1 must be equal to Q_2; therefore,

$$K_1 a \frac{dh_1}{dl_1} = K_2 c \frac{dh_2}{dl_2} \tag{4.51}$$

Since the head loss between the two equipotential lines is the same in both strata, $dh_1 = dh_2$ and we can divide both sides of Equation 4.51 by dh_1:

$$K_1 \frac{a}{dl_1} = K_2 \frac{c}{dl_2} \tag{4.52}$$

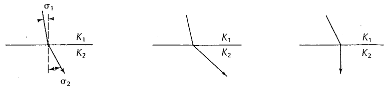
From the geometry of the triangles, $a = b \cos \sigma_1$ and $c = b \cos \sigma_2$. Furthermore, $b/dl_1 = 1/\sin \sigma_1$ and $b/dl_2 = 1/\sin \sigma_2$. Substituting these into Equation 4.52, we obtain

$$K_1 \frac{\cos \sigma_1}{\sin \sigma_1} = K_2 \frac{\cos \sigma_2}{\sin \sigma_2} \tag{4.53}$$

Since $\tan \sigma = (\sin \sigma)/(\cos \sigma)$, Equation 4.53 can be rewritten as the tangent law of refraction.

$$\boxed{\frac{K_1}{K_2} = \frac{\tan \sigma_1}{\tan \sigma_2}} \tag{4.54}$$

Refraction of Flowline



▲ FIGURE 4.14
A. Refraction of a flowline crossing a conductivity boundary. **B.** Refracted flowline going from a region of low to high conductivity. **C.** Refracted flowline going from a region of high to low conductivity.

Refraction of Flowline

► FIGURE 4.15
A flow net with flow crossing a conductivity boundary showing refraction of flowlines and equipotential lines. The hydraulic conductivity above the boundary is less than that below the boundary.