SURFACE WAVES

Content

(From Lay and Wallace,1995; Ikelle and Amundsen, 2005)

INTRODUCTION

Surface waves are boundary waves, "skin waves" or evanescent waves, that is waves which travel associated to a boundary between different earth materials but whose displacement amplitude decays exponentially away from the same boundary in both or only one direction.

Different types of surface waves exist under certain conditions: (1) Below a free surface (**Rayleigh Wave;** Rayleigh, 1894**),** just (2) above and below a layer-to-layer contact (**Stonely Wave,** Stonely,), and (3) just above and below the contact between a fluid and solid layer (**Scholte Wave**). Strictly speaking Love waves are not surface waves. Love waves are trapped waves travelling inside a layer bounded via post-critical reflection to between the upper and lower boundaries of hat layer. This layer is bounded above by a free surface and below by a solid.

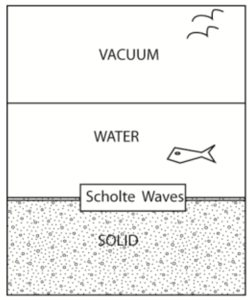
When a surface wave has a wavelength that approaches that of the whole object in which it is traveling such as an iron beam, the crustal layer of the earth, a man-made levee we refer to these waves as normal modes, or resonating vibrations. When the object is much larger than the wavelength under consideration we are able to ignore these resonating oscillations.

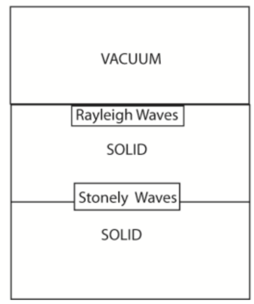
Ikelle and Amundsen (2005) treat Rayleigh waves as a particular case of a Scholte wave when the water/fluid layer thins to nothing.

Instead, we will derive the Rayleigh wave solution to the wave equation following Lay and Wallace (2005), which I believe is a more didactic if less general approach to understanding surface waves.

Using and understanding Rayleigh wave propagation required historically a blend of intuition to make the right approximation for the boundary conditions as well mathematical labor to test for different cases and arrive at usable solutions.

At a free surface, Snell's Law still applies and an impinging P-wave from below will reflect (except at normal conditions) as a combined P and SV wave. Similarly an SV wave will bounce from below as a SV and converted P wave. When critical angles are reached from below the free surface and beyond these critical angles the boundary conditions require that the converted modes in both cases are not physically real. However, the converted modes can co-exist, fulfill boundary conditions and generate complex elliptical particle motions.





Free Surface: A boundary across which shear stresses disappear and the displacements vanish as well.

$$\sigma_{ij}^{+} = \sigma_{ij}^{-} \text{ and } \sigma_{ij}^{+} = 0$$

$$u_{ij}^{+} = u_{ij}^{-} \text{ and } u_{zz}^{+} = 0$$

Example of a Free Surface: A boundary between the earth and air.

Snell's Law: p (horizontal component of slowness) is conserved in a horizontally layered earth:

$$p^2 + \eta^2 = \frac{1}{|\vec{V}|^2} = |\vec{s}^{\,2}|$$

$$\eta = \frac{cos\,\theta}{|\vec{V}|}$$

$$p^2 + \eta^2 = \frac{1}{|\vec{V}|^2} = |\vec{s}^{\,2}|$$

$\vec{s} = \vec{s}\,(p,\eta)$ where s is the slowness vector.

$\vec{k} = \frac{\omega}{\vec{V}}$, where k is the wavenumber vector

$$k_x = p\omega$$
$$k_z = \eta\omega$$

We will see that there are situations at critical angle :

$$\theta_{critical} = sin^{-1}\left(\frac{1}{V_P}\right),$$

where

$$p = \frac{\sin\theta_c}{V_P} = \frac{\sin(\pi/2)}{V_P} = \frac{1}{V_P}$$
$$\eta = \frac{\cos\theta_c}{V_P} = \sqrt{\frac{1-\sin^2\theta_c}{V_P^2}}$$
$$= \sqrt{\frac{1}{V_P^2} - p^2}$$

In the free-surface case of a wave arriving from below when the critical angle is exceeded:

$$p > \frac{1}{V_P}$$
$$\eta_P = \sqrt{(<0)}$$
$$\eta_P = i\eta_{P_{im}}$$

and the vertical slowness parameter becomes imaginary!

Case 1: Upgoing P wave at a free boundary

When the upgoing wave is a P wave at the free boundary or free surface we can describe the displacement potentials as follows:

$$\chi = \chi_{incident\ P-wave} + \chi_{reflected\ P-wave}$$

$$\chi = Ae^{i\omega(px_1 - \eta_P x_3 - t)} + Be^{i\omega(px_1 + \eta_P x_3 - t)}$$

The displacement potential χ represents all the P-wave displacements.

As for the SV-wave that is created during the reflection at angles other than 90 degrees,

we can describe its displacement potential by:

$$\psi_2 = \psi_{SV-wave\ reflection}$$
$$\psi_2 = Ce^{i\omega(px_1 + \eta_S x_3 - t)}$$

We can show that these types of potentials lead to the following solutions for the plane wave reflection coefficients:

$$R_{PP} = \frac{B}{A} = \frac{\left[(\lambda + 2\mu)\eta_P^2 + p^2\lambda\right] + \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_S^2\right)}}{-\left[(\lambda + 2\mu)\eta_P^2 + p^2\lambda\right] + \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_S^2\right)}}$$

and $$R_{PS} = \frac{C}{A} = \left(\frac{4\mu\eta_P}{p^2 - \eta_S^2}\right) \frac{(\lambda + 2\mu)\eta_P^2 + p^2\lambda + \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_S^2\right)}}{(\lambda + 2\mu)\eta_P^2 + p^2\lambda - \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_S^2\right)}}$$

Exercise1 : These two solutions above were obtained by considering that at the free surface $\sigma_{33} = 0$. Use the potentials to show that the following is true:

$$(A + B)\left[(\lambda + 2\mu)\eta_P^2 + p^2\lambda\right] + C\left(2\mu p \eta_S\right) = 0$$ (Equation 4.3 in Lay and Wallace, 2005)

These solutions also require that we consider $\sigma_{13} = 0$ at a free suface. Use the potentials above to show that the following is true:

$$(A - B)2p\eta_P - C\left(p^2 - \eta_S^2\right) = 0$$ (Equation 4.4 in Lay and Wallace, 2005)

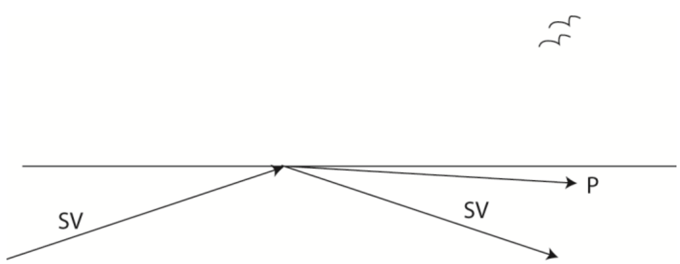
Exercise 2 : In the solutions shown above (i.e. for R_{PS} and R_{PP}) we expect no reflections of SV waves for a P-wave incident angle of 90 degrees. Show whether this true or not.

Exercise 3. Also in the solutions shown above there are two cases when all the reflected potential amplitudes are in SV form. Find out the two incident angles at which this phenomenon occurs. You can use the notes I placed on moodle from Lay and Wallace (2005)

The amplitudes of the reflecting P and converted SV waves can be estimated. One prediction derived from the study of the converted SV waves is that at grazing angles of incidence the SV waves return no energy to the solid medium.

Case 2: Upgoing SV wave at a free boundary

In the case of an upgoing SV wave we expect, in general, to see a downgoing SV wave as well as a downgoing P wave.



A special case exists (critical angle) where the P wave travels only along the boundary. This angle is

$$\theta_{critical} = sin^{-1}\left(\frac{V_S}{V_P}\right)$$

If the angle of incidence exceeds this critical value, the reflected angle becomes complex and the vertical component of slowness becomes purely imaginary. In this situation, the potential χ experiences a phase shift (time delay) and its amplitude decays exponentially with distance below the surface.

However, it can be shown that there is no physical energy returned to the lower half-space by the converted P wave.

When the upgoing wave is a SV wave at the free boundary or free surface we can describe the displacement potentials as follows:

$$\chi = \chi_{reflected\ but\ converted\ P-wave}$$

$$\chi = Fe^{i\omega(px_1 - \eta_P x_3 - t)}$$

Exercise 4: I have intentionally left a mistake in the last equation above. What is wrong with one of the exponent terms and why must it be otherwise?

The displacement potential χ represents all the P-wave displacements.

As for the SV-wave that is created during the reflection, we can describe its displacement potential by:

$$\psi_2 = \psi_{SV-wave\ reflection}$$
$$\psi_2 = De^{i\omega(px_1 - \eta_P x_3 - t)} + Ee^{i\omega(px_1 + \eta_P x_3 - t)}$$

Again, it is not immediately obvious why the potential would have these forms except that because of thorough mathematical analysis over the centuries and by development of electromagnetic theory similar solutions were derived, tested and eventually borrowed by seismologists.

We can show that these types of potentials lead to the following solutions for the plane wave reflection coefficients:

$$R_{SS} = \frac{E}{D} = \frac{\left[(\lambda+2\mu)\eta_P^2 + p^2\lambda\right] + \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_P^2\right)}}{-\left[(\lambda+2\mu)\eta_P^2 + p^2\lambda\right] + \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_B^2\right)}}$$

(Equation 4.8 from Lay and Wallace, 2005)

and $$R_{SP} = \frac{F}{D} = \frac{4\mu\eta_P}{(\lambda+2\mu)\eta_P^2 + p^2\lambda - \dfrac{4\mu p^2 \eta_P \eta_S}{\left(p^2 - \eta_S^2\right)}}$$

(Equation 4.9 from Lay and Wallace, 2005)

Exercise5 : Plot R_{SP} using any program of your choice for values of V_P

=1600 m/s and V_S=1000 m/s and density=1600 kg/m^3

=

Rayleigh Waves

Rayleigh waves occur only when critical and post-critical P and SV waves travel simultaneously along the free surface. Individually they can not exist, but together they can. That is, the two previous cases can be used to determine the angles when this occurs. Rayleigh wave propagation speeds are found to be less than the speed of a shear wave in the media. For a Poisson solid half-space and for typical values of Poisson's ratio (0.2 – 0.4,) the Rayleigh wave velocity is 0.9 – 0.95 of the Shear wave velocity.

Following Lay and Wallace (2005) let's see how two evanescent waves can co-exist mathematically (at conditions that are at or post-critical). Very astutely, Rayleigh (1887) saw that(1) if we allow the propagation velocity along the surface to be such that:

$$\textit{horizontal apparent velocity}: \frac{1}{p} = V_R < V_S < V_P$$

And (2) if we have both evanescent waves propagating simultaneously, then, the first exponent terms on the right hand sides of the following potentials decay exponentially as a function of distance

$$\begin{aligned}\chi &= Ae^{i\omega(px_1+\eta_P x_3-t)} \\ &= Ae^{(-\omega x_3)\eta_{P_{im}}} e^{i\omega(px_1-t)}\end{aligned}$$

$$\begin{aligned}\varphi &= Be^{i\omega(px_1+\eta_S x_3-t)} \\ &= Be^{(-\omega x_3)\eta_{S_{im}}} e^{i\omega(px_1-t)}\end{aligned}$$

I emphasize that the above only occurs the special velocity requirements shown above, so that a negative value can appear in the numerator:

$$\eta_P = \sqrt{\frac{1}{V_P^2} - p^2}$$

$$\eta_P = \sqrt{\frac{1}{V_P^2} - \frac{1}{V_R^2}} = \sqrt{(<0)}$$

$$\eta_P = i\sqrt{\frac{1}{V_R^2} - \frac{1}{V_P^2}}$$

$$\eta_P = i\eta_{P_{im}}$$

Exercise 4:

For a Rayleigh wave the most famous aspect is its retrograde elliptical motion. Let's show that is true. Take a Poisson solid with

Rayleigh wave velocity = 0.919 V_S

And Shear wave velocity = 0.531 V_P.

Then:

$$u_1 = -Ak\,sin\left(kx_1 - \omega t\right)\left[e^{(-0.85kx_3)} - 0.58e^{-0.39kx_3}\right]$$

And

$$u_3 = -Ak\,cos\left(kx_1 - \omega t\right)\left[e^{(-0.85kx_3)} - 1.47e^{-0.39kx_3}\right]$$

Graph the retrograde elliptical motion at the surface for a case where A=1,

Frequency=10 Hz, and V_P=200 m/s

Also estimate at what depth the amplitude goes to 50 % .

Can you detect at what depth the motion becomes prograde elliptical instead of retrograde elliptical?