
ACOUSTIC WAVE EQUATION

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INTRODUCTION

As we try to visualize the earth seismically we make certain physical simplifications that make it easier to make and explain our observations. Liner (2004, p.33) considers three types of simplified descriptions for the earth materials: fluid, porous and solid.

Fluids, e.g. gases and liquids characteristically can deform only by compression and decompression. Only *acoustic*, or *sound*, or *P-wave* or *compressional body waves* are the natural wave vibrations that can propagate through fluid earth materials.

In solids we can simplify P-waves via an acoustic model so as to ignore the effect of coupling to shear stresses (p.55, Ilke and Amundsen). Note that in this case the shear modulus still remains as one of the Lamé's parameters. For liquids the shear modulus is 0 and for solids it is non-zero.

In a fluid, seismic data can be collected by hydrophones in the form of pressure measurements. The pressure field is a scalar field, which is simpler to deal with mathematically, than a tensor field.

From the following mathematical derivations we can reach several accurate concepts to help us visualize particle strain during acoustic wave transmission in a fluid.

- First, within a fluid, and at a given point, particle motion increases as the pressure gradient increases.
- Second, the larger the particle density the slower the particle acceleration. Note that an unweathered piece of granite is denser than a piece of slate from a blackboard and so based ON ONLY the property of its density particle acceleration will be smaller in granite than in slate. [\[Q\]](#)
- Third, the more incompressible the fluid the faster the particle motion. A fast particle motion is related with a fast transmission of mechanical vibration through the fluid. If we use the [ball and spring](#) model this implies that as we substitute stiffer and stiffer springs in the picture particle motion is faster and the vibration or wave is able to cross between balls at a faster rate ("speed of sound").

On the Acoustic Wave Equation

Most people would say the speed of sound increases as the density of the material that sounds travels through. For example, sound waves travel faster in the water of the swimming pool than by shouting above the water. Yet, the acoustic equation of motion implies the opposite. Explain why the speed of sound is so much greater at the center of the earth than in the crust near the surface of the earth in terms of the acoustic wave equation and its physical implications.

On Hydrophones versus Geophones

Q. 1 On land seismic surveys data is collected by devices that are able to convert ground velocity or acceleration into a voltage. In marine settings seismic streamers tow only hydrophones which are arrays of pressure transducers. Since acoustic waves produce particle motion in fluids shouldn't we be able to use 3-component geophones in fluids as well as pressure transducers? Think through and explain why you think hydrophones are the choice for marine acquisition work.

Q.2 A seismic acquisition company is currently marketing digital 3-component accelerometers as a substitute for 3-component geophones. What is the advantage of using an accelerometer over a geophone for a seismic land-based survey?

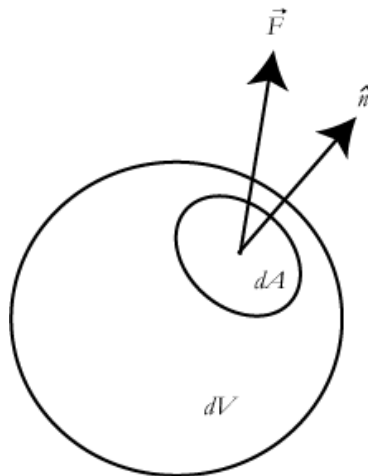
Mathematical Derivation

We saw in the section on [tensors](#) that

$$\mathbf{T}_i = \mathbf{n}_j \sigma_{ji}$$

The total force $\vec{F} = (F_1, F_2, F_3)$ exerted by the medium on to the volume V through the small surface given area dA is

$$F_i = \sigma_{ji} n_j dA \quad \text{where} \quad \sigma_{ji} n_j \text{ is force per unit area (pressure)} \quad (1)$$



For example in a fluid:

$$\sigma_{ij} = -P \delta_{ij}$$

(2)

where P is pressure and where compression is by convention negative.

Expression (2) can also be expressed as

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = - \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{bmatrix}$$

$$= - \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that we can view the **Kronecker delta** as a second order tensor where there **are NO off-**main-diagonal components because there is no shear, i.e.

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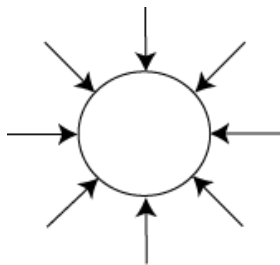
$$\sigma_{ij} = \sigma_{ji}$$

So now combining (1) and (2) we have

$$\begin{aligned} F_i &= -P \delta_{ij} n_j dA \\ &= -P n_i dA \end{aligned}$$

that is, in vectorial notation, $\vec{F} = -P \hat{n} dA$.

If there is no gradient in the pressure there is no net force acting on it. For example a neutrally buoyant sphere will neither rise nor fall immersed in a fluid.



BULK MODULUS AND LAMÉ'S PARAMETERS

We aim to show that how the bulk modulus is related to Lamé's parameters

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Previously, in dealing with the elastic wave equation we saw that Hooke's law for the case of an isotropic, heterogeneous medium took on the form

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

and the scalar expressions that related the strain field to the gradient of the displacement field or the dilatation were as:

$$e_{kk} = u_{k,k} = \nabla \cdot \vec{u} = \frac{\Delta V}{V}$$

(The volumetric strain is the same as the sum of the linear strains in each of the principal directions if the deformation is small, as in the case of ideal elasticity. (Proof of this relation is given to the reader to verify, until we demonstrate this approximation in class)

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For a hydrostatic pressure field where $\sigma_{11} = \sigma_{22} = \sigma_{33} = -P$ we can rephrase Hooke's Law, substituting (2) to obtain :

$$-P \delta_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \quad (3)$$

We contract the indices, making $i = j$ in order to consider only the non-zero contributions to the sum:

$$\begin{aligned} -P \delta_{ii} &= \lambda \delta_{ii} e_{kk} + 2\mu e_{ii} \\ -P \cdot 3 &= \lambda \cdot 3 \cdot e_{kk} + 2\mu e_{kk} \quad (\text{Note that } e_{kk} \equiv e_{ii}) \\ -P \cdot 3 &= (\lambda \cdot 3 + 2\mu) e_{kk} \end{aligned}$$

By replacing terms with other equivalent expressions, noted immediately above,

$$\begin{aligned} -P \cdot 3 &= (\lambda \cdot 3 + 2\mu) \frac{\Delta V}{V} \\ P &= -\left(\frac{\lambda \cdot 3 + 2\mu}{3}\right) \frac{\Delta V}{V} \\ P &= -\left(\lambda + \frac{2}{3}\mu\right) \frac{\Delta V}{V} \\ -P \frac{V}{\Delta V} &= \lambda + \frac{2}{3}\mu \end{aligned}$$

In this form, we can show that the bulk modulus ($k = -\frac{\Delta P}{\frac{\Delta V}{V}}$) can be expressed in terms of

Lamé's parameters:

$$k = \lambda + \frac{2}{3}\mu$$

In the acoustic case, we have that $\mu = 0$ and $k = \lambda$, so that equation (3) can be re-expressed as

$$\begin{aligned}
 -P\delta_{ij} &= \kappa\delta_{ij}e_{kk} \\
 -P &= \kappa e_{kk} \\
 &= k\nabla\cdot\vec{u} = k\frac{\Delta V}{V}
 \end{aligned} \tag{4}$$

from which we see that the divergence of the displacement field is proportional to the pressure.

Particle acceleration and its relation to density the pressure field and wave velocity

We can predict the different parameters in the [equation of motion](#):

$$\begin{aligned}
 \rho\vec{\ddot{u}} &= \sigma_{ij,j} \\
 &= (-P\delta_{ij})_{,j} \\
 &= -(P_{,j}\delta_{ij} + P\delta_{ij,j}) \text{ (non-zero components only exist for } j=i) \\
 &= -P_{,j}\delta_{ij} = -P_{,i}\delta_{ii} \\
 &= -P_{,i} \equiv -\frac{\partial P}{\partial x_i}
 \end{aligned}$$

In vectorial notation we can also express this as

$$\begin{aligned}
 \rho\vec{\ddot{u}} &= -\nabla P \\
 \vec{\ddot{u}} &= -\frac{\nabla P}{\rho}
 \end{aligned} \tag{5}$$

In this form the equation of motion tells us that the particle acceleration in a body increases with larger gradients in the pressure field but decreases as the material becomes denser and requires more energy to move.

We can also estimate how pressure changes in space can affect the particle acceleration, by taking the divergence of the above expression.

$$\begin{aligned}
 -\nabla\cdot(\nabla P) &= \nabla\cdot(\rho\vec{\ddot{u}}) \\
 -\nabla^2 P &= \rho(\nabla\cdot\vec{\ddot{u}}) + \vec{\ddot{u}}\cdot\nabla\rho \\
 &= \rho\frac{\partial^2}{\partial t^2}(\nabla\cdot\vec{u}) + \vec{\ddot{u}}\cdot\nabla\rho
 \end{aligned}$$

