Tensors are abstract objects describable by **arrays of functions**. Each function of such an array is also called a **component.** Components are functions of the selected co-ordinate system (Hadsell, 1995). As such, the component can change in aspect with the change of reference system. However, the value of the property they serve to represent does not.

Let's try to visualize the components of a tensor and their relation to real physical values they represent:

A tensor is called an **nth order tensor** when it comprises an array of r^n components, where *r* is the number of the dimension (2D, 3D, etc.) and *n* is called the order We will be mainly seeing second or third-order tensors in three dimensions, so that our arrays can have from 3^2 to 3^3 components. Arrays with nine components can be written in the form of a 3x3 matrix. Arrays with 81 components are more difficult to visualize but we can use the Voigt notation in this case. In the case of more than two dimensions indicial notation becomes a very convenient way to deal with tensors.

As an example can use the Levi-Civita tensor of order 3, where *i,j and k* can equal 1,2,3:
ε_{ijk}

In this example we can think of the *i* as the row number that contains three, 3x1 vectors.

$$\varepsilon_{ijk} = \begin{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\2\\1\end{pmatrix} & \begin{pmatrix}1\\3\\1\end{pmatrix} \\ \begin{pmatrix}2\\1\\1\end{pmatrix} & \begin{pmatrix}2\\2\\1\end{pmatrix} & \begin{pmatrix}2\\3\\1\end{pmatrix} \\ \begin{pmatrix}3\\1\\1\end{pmatrix} & \begin{pmatrix}3\\2\\1\end{pmatrix} & \begin{pmatrix}3\\3\\1\end{pmatrix} \end{pmatrix}$$

for i=1,2,3; j=1,2,3; k=1

$$\varepsilon_{ijk} = \begin{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\2\\1\end{pmatrix} & \begin{pmatrix}1\\3\\1\end{pmatrix} \\ \begin{pmatrix}2\\1\\1\end{pmatrix} & \begin{pmatrix}2\\2\\1\end{pmatrix} & \begin{pmatrix}2\\3\\1\end{pmatrix} \\ \begin{pmatrix}3\\1\\1\end{pmatrix} & \begin{pmatrix}3\\2\\1\end{pmatrix} & \begin{pmatrix}3\\3\\1\end{pmatrix} \end{pmatrix}$$

for i=1,2,3; j=1,2,3; k=2

$$\varepsilon_{ijk} = \begin{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\2\\1\end{pmatrix} & \begin{pmatrix}1\\3\\1\end{pmatrix} \\ \begin{pmatrix}2\\1\\1\end{pmatrix} & \begin{pmatrix}2\\2\\1\end{pmatrix} & \begin{pmatrix}2\\3\\1\end{pmatrix} \\ \begin{pmatrix}3\\1\\1\end{pmatrix} & \begin{pmatrix}3\\2\\1\end{pmatrix} & \begin{pmatrix}3\\3\\1\end{pmatrix} \end{pmatrix}$$

for i=1,2,3; j=1,2,3; k=3

We can consider j as the column and k as the layer or depth of the 27-component system.

$$\varepsilon_{ijk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\varepsilon_{ijk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\varepsilon_{ijk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

STRESS TENSOR

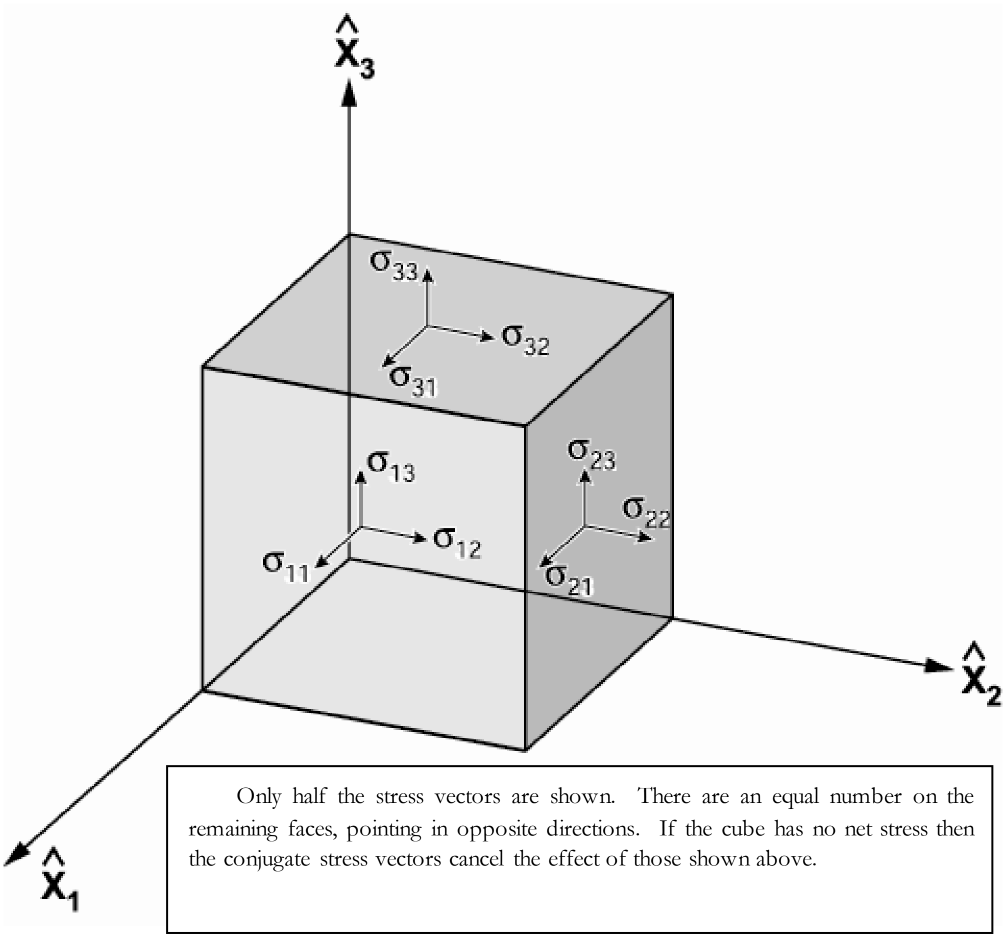
A stress tensor is of order [illegible] e [illegible] er words, at one point in three-[illegible] be [illegible] ng three vectors centered at the sa[illegible]

In order to study and define the stress at a point we can study stress in two ways. First, on three orthogonal planes passing through the point, or alternatively, we can study stress on the six sides of an infinitesimally small cube as its volume in the limit, diminishes in size to nothing.

At a finite size, this cube as being acted upon by body forces on all the particles of matter it contains. Body forces act at a distance and could include a gravitational, electrical or magnetic field that is acting on the particles in the cube. If we assume that the body is not experiencing any linear acceleration which can contribute to the gradient of the stress vector field then we can disregard these body forces.

Also, if there is any net torque acting on the faces of the cube, in the limit as the volume is taken to a point, for a fixed stress the body would spin faster and faster. It is easier to see this small cube as non-rotating, with no net torque acting upon it.

In a Eulerian view of the world we fix our point of observation, say at a geophone and measure how quickly the ground moves with respect to the fixed point. In many cases and in the following approach we will use a Eulerian view of the world. Note that in other cases, it is easier mathematically to fix our reference frame upon the particle in motion and describe the wavefield from the point of view travelling with the wave (Lagrangian view).



Only half the stress vectors are shown. There are an equal number on the remaining faces, pointing in opposite directions. If the cube has no net stress then the conjugate stress vectors cancel the effect of those shown above.

Each face of the cube can experience traction or stress. We take as convention that the surface stress acts on the outer surfaces of the cube upon the inner surface. Stress on a surface can be treated as a "stress vector" and is also called a **traction vector ($\vec{T}$)**, where, in the limit, traction equals force per unit area at a point on one face in the direction normal to the face $(\hat{n})$:

$$\vec{T}(\hat{n}) = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$$

$$T_j(n_i) = \lim_{\Delta A_i \to 0} \frac{\Delta F_j}{\Delta A_i}$$

$$\sigma_{ij} = \lim_{\Delta A_i \to 0} \frac{\Delta F_j}{\Delta A_i}$$

Using indicial notation, $\Delta\vec{F}$ can be written as $F_j = \Delta F_j \hat{x}_j$. That is, the force on a particular face, as the face becomes very small, has three components. For example, the traction or stress vector according to the definition above, and for a particular face (n_1) is as follows:

$$T_j(n_1) = \lim_{\Delta A_1 \to 0} \frac{\Delta F_j}{\Delta A_1}$$
$$\sigma_{1j} = \lim_{\Delta A_1 \to 0} \frac{\Delta F_j}{\Delta A_1}$$

That is, oneach face there are three stress vectors to represent the action of force in three directions as the area becomes very small.

We use the following convention to denote stress:

$$\sigma_{ij}$$

where i is the index of the axis to which the face is normal and j is the index of the direction in which the component of the traction vector is applied. When the component of the traction vector is applied in the direction of the basis vector, our convention is to use a positive value and negative when the component is applied opposite to the sense of the basis vector. Sometimes

$\sigma_{ij} = \sigma_{(i)i}$ and each component is called normal traction component. e.g.,

$$\sigma_1, \sigma_2, \sigma_3$$

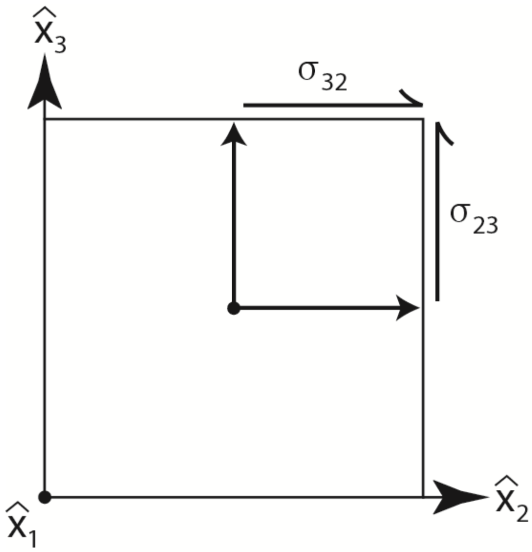
Often, when $i \neq j$, $\sigma_{ij} = \tau_{ij}$ and , τ_{ij} being used to denote the shear traction component. Ikelle and Amundsen (2005) use τ_{ij} for both normal and shear traction components.

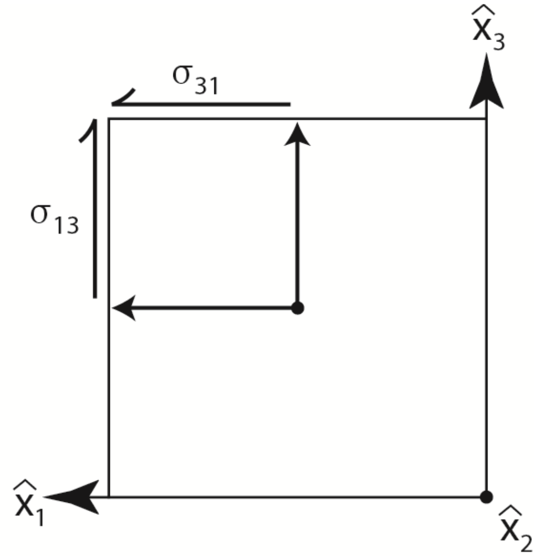
If the cube does not rotate then the shear tractions must cancel each other out, so that for each case, the net torque is zero, where

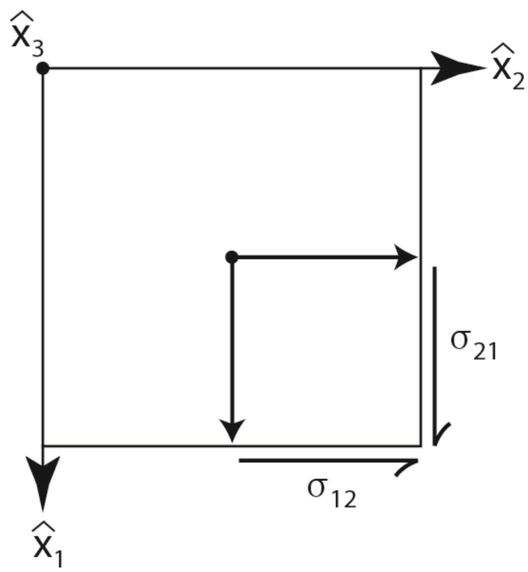
$$\boldsymbol{Torque} = \boldsymbol{r} \times \boldsymbol{F}$$

$$|Torque| = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta$$

We obtain force from stress by multiplying the stress over the area it acts. Force is the product of traction times the surface area over which the traction is applied.







These three plots represent the orthogonal views of the stress cube down each of the major axes. From left to right and top-to-bottom, we are looking down axis 1,axis 2, and axis 3 respectively.

If the cube in question does not experience a net rotation, the net torque should be 0.

Of the 18 traction components, (3 traction vectors on 6 faces of the cube) there are three pairs that must cancel each other's effects if we assume that there is no net rotation during the application of stress. For the <u>three</u> cases we get:

$$\hat{x}_1\left(\boldsymbol{\sigma}_{21}dx_1dx_3\right)\times r\hat{x}_2 = -\hat{x}_2\left(\boldsymbol{\sigma}_{12}dx_2dx_3\right)\times r\hat{x}_1$$

$$\hat{x}_3\left(\boldsymbol{\sigma}_{23}dx_3dx_1\right)\times r\hat{x}_2 = -\hat{x}_2\left(\boldsymbol{\sigma}_{32}dx_2dx_1\right)\times r\hat{x}_3$$

$$\hat{x}_3\left(\boldsymbol{\sigma}_{13}dx_3dx_2\right)\times r\hat{x}_1 = -\hat{x}_1\left(\boldsymbol{\sigma}_{31}dx_1dx_2\right)\times r\hat{x}_3$$

In indicial notation these three cases can be written as:

$$\varepsilon_{ijk}\left(\boldsymbol{\sigma}_{21}dx_1dx_3,0,0\right)_j(0,r,0)_k=-\varepsilon_{ijk}\left(0,\boldsymbol{\sigma}_{12}dx_2dx_3,0\right)_j(r,0,0)_k$$

$$\varepsilon_{ijk}\left(0,0,\boldsymbol{\sigma}_{23}dx_3dx_1\right)_j(0,r,0)_k=-\varepsilon_{ijk}\left(0,\boldsymbol{\sigma}_{32}dx_2dx_1,0\right)_j(0,0,r)_k$$

$$\varepsilon_{ijk}\left(0,0,\boldsymbol{\sigma}_{13}dx_3dx_2\right)_j(r,0,0)_k=-\varepsilon_{ijk}\left(\boldsymbol{\sigma}_{31}dx_1dx_2,0,0\right)_j(0,0,r)_k$$

For the equation to hold true then $\sigma_{ji}=\sigma_{ij}$, which is the description of a **symmetric tensor.** In other words, in order for the net torque to be zero the tensor must be symmetric.

A tensor, t_{ij} is symmetric iff

$$\varepsilon_{ijk}t_{jk}=0.$$

The symmetry means that the off-diagonal terms are equal

$$\begin{aligned}\varepsilon_{ijk}t_{jk}&=\sum_{j=1}^{j=3}\varepsilon_{ijk}t_{jk}\\&=\sum_{k=1}^{k=3}\left(\varepsilon_{i1k}t_{1k}+\varepsilon_{i2k}t_{2k}+\varepsilon_{i3k}t_{3k}\right)\\&=\varepsilon_{i11}t_{11}+\varepsilon_{i12}t_{12}+\varepsilon_{i13}t_{13}+\\&\quad\varepsilon_{i21}t_{21}+\varepsilon_{i22}t_{22}+\varepsilon_{i23}t_{23}+\\&\quad\varepsilon_{i31}t_{31}+\varepsilon_{i32}t_{32}+\varepsilon_{i33}t_{33}\end{aligned}$$

We can show that the above condition is true by examining all the cases of I (=1,2,3). We can start by seeing if the example is true when *i=1:*

$$\begin{aligned}
\varepsilon_{1jk}t_{jk} &= \sum_{j=1}^{j=3}\varepsilon_{1jk}t_{jk} \\
&= \sum_{k=1}^{k=3}\left(\varepsilon_{11k}t_{1k} + \varepsilon_{12k}t_{2k} + \varepsilon_{13k}t_{3k}\right) \\
&= \varepsilon_{111}t_{11} + \varepsilon_{112}t_{12} + \varepsilon_{113}t_{13} + \\
&\quad \varepsilon_{121}t_{21} + \varepsilon_{122}t_{22} + \varepsilon_{123}t_{23} + \\
&\quad \varepsilon_{131}t_{31} + \varepsilon_{132}t_{32} + \varepsilon_{133}t_{33} \\
&= 0 + 0 + 0 + \\
&\quad 0 + 0 + t_{23} + \\
&\quad 0 - t_{32} + 0
\end{aligned}$$

Remember that

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

Recall: An essential working assumption for this theorem is that the body must be in mechanical equilibrium, that is, the body is not experiencing any change in its linear or angular momentum.

CAUCHY'S THEOREM OR STRESS PRINCIPLE (BEN- MENAHEM AND SINGH, 2000)

Cauchy's theorem states that given a plane of interest, we can define a traction vector (a first-order tensor) at a point ($\vec{T}(\vec{n})$) on this plane in terms of any orthonormal reference system. That means that although the stress field tensor is a second-order tensor we can deal with stress as a lower-order tensor, simplifying our mathematical complexity. According to Cauchy's theorem and using indicial notation we can have

$$\boldsymbol{T}_i = \sigma_{ji}\boldsymbol{n}_j$$

In other words, stress at a point can be manipulated as if it were only a vector, without loss of accuracy.

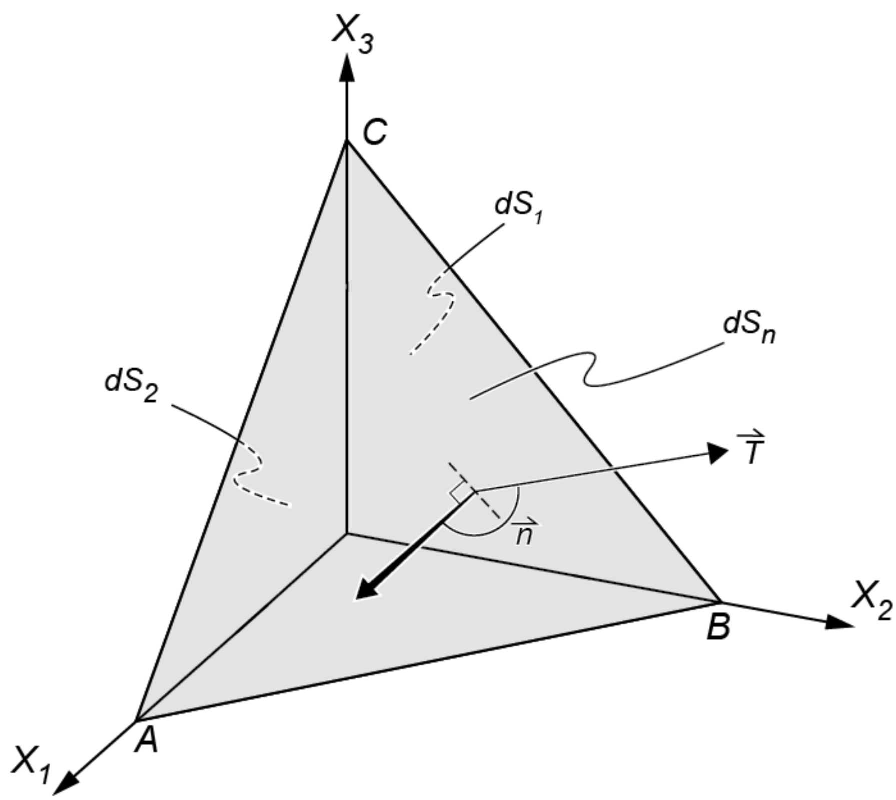
Especially note the presence of the components $\boldsymbol{n}_j$. The Cauchy theorem requires that the traction vector be a function of a plane, which is described by the direction cosine components (n_j) of the **unit** vector ($\vec{n}$) normal to the plane. In summary:

$$\boldsymbol{n}_j = cos\,\theta_j$$

$$\boldsymbol{n}_j \boldsymbol{n}_j = 1 = \left(cos\,\theta_j\right)\left(cos\,\theta_j\right)$$

The angle θ is the angle between each component of the unit vector $\boldsymbol{n}$ and the corresponding coordinate axis.

We can demonstrate Cauchy's theorem by balancing the forces on the sloping face of a tetrahedron against the forces on the other three sides. This tetrahedron in the limit goes to a point as its dimensions become infinitesimally small. If the equilibrium condition is met (no net torque) we will discover that Cauchy's theorem holds true.



It will be very helpful to note before we begin that

$$dS_i = dS_n\left(\cos\theta_i\right),$$

e.g., $$dS_1 = dS_n\left[cos\left(\boldsymbol{n}\ ,\hat{x}_1\right)_1\right]$$

where $\cos\theta_i$ is the direction cosine with regard to the $\hat{x}_i$ basis vector.

We begin by balancing forces:

$$\int \vec{T}(\vec{n})dS = \vec{T}(\vec{n})dS_n + \vec{T}(-\hat{x}_1)dS_1 + \vec{T}(-\hat{x}_2)dS_2 + \vec{T}(-\hat{x}_3)dS_3$$

$$= \vec{T}(\vec{n})dS_n - \vec{T}(\hat{x}_1)dS_1 - \vec{T}(\hat{x}_2)dS_2 - \vec{T}(\hat{x}_3)dS_3$$

Note that $\vec{T}(\hat{x}_1)$ is a three-component stress vector across the x_1 plane, which has a normal $\hat{x}_1$, and is multiplied by dS_1 a scalar (units of area), so that each term on the right hand side is also a vector scaled by the value of the surface across which the stress vector is acting.

Given that the tetrahedron is in equilibrium, then $\int \vec{T}(\vec{n})dS = 0$, so that the first term on the right-hand-side must equal the sum of the remaining three, i.e.

$$\vec{T}(\vec{n})dS_n = \vec{T}(\hat{x}_1)dS_1 + \vec{T}(\hat{x}_2)dS_2 + \vec{T}(\hat{x}_3)dS_3 .$$

In other words, if we know the stress on three planes of the tetrahedron (RHS) we can tell the stress on a general plane (LHS)!

By substituting the scalar value of the surface area of each side of the tetrahedron (dS_i) in terms of the surface area of the main face dS_n, we have:

$$\vec{T}(\vec{n})dS_n = \vec{T}(\hat{x}_1)dS_n\left(\frac{\vec{n}\cdot\hat{x}_1}{|\vec{n}|}\right) + \vec{T}(\hat{x}_2)dS_n\left(\frac{\vec{n}\cdot\hat{x}_2}{|\vec{n}|}\right) + \vec{T}(\hat{x}_3)dS_n\left(\frac{\vec{n}\cdot\hat{x}_3}{|\vec{n}|}\right)$$

Rearranging terms, we have:

$$\vec{T}(\vec{n})dS_n = \frac{\vec{n}}{|\vec{n}|}\left(\vec{T}(\hat{x}_1)\bullet\hat{x}_1 + \vec{T}(\hat{x}_2)\bullet\hat{x}_2 + \vec{T}(\hat{x}_3)\bullet\hat{x}_3\right)dS_n$$

Dividing both sides by dS_n, we have:

$$\boldsymbol{T}_i(\vec{n}) = \frac{\boldsymbol{n}_i}{|\vec{n}|}T_j(\hat{x}_i)\hat{x}_j$$

Each component of the general stress tensor on the left-hand side of the equation comprises components of stress of all three other faces.

As an example, the first component of the stress vector in the $\hat{x}_1$ direction on a face perpendicular to the $\hat{x}_2$ direction is $T_1(\hat{x}_2)\hat{x}_1$, or σ_{21}. The first component of stress vector in the $\hat{x}_2$ direction on a face perpendicular to the $\hat{x}_2$ direction is $T_1(\hat{x}_2)\hat{x}_2$, or σ_{22}. The first component of the stress vector in the $\hat{x}_3$ direction on a face perpendicular to the $\hat{x}_2$ direction is $T_1(\hat{x}_2)\hat{x}_3$, or σ_{23}.

The complete stress vector on this face has three components:

$$\begin{aligned}\boldsymbol{T}_2(\vec{n}) &= \vec{T}(\hat{x}_2) \\ &= \boldsymbol{n}_2\left(T_1(\hat{x}_2)\hat{x}_1 + T_2(\hat{x}_2)\hat{x}_2 + T_3(\hat{x}_2)\hat{x}_3\right)\end{aligned}, \text{ or}$$

$$\vec{T}(\hat{x}_2) = \boldsymbol{n}_2\left(\sigma_{21}\hat{x}_1 + \sigma_{22}\hat{x}_2 + \sigma_{23}\hat{x}_3\right)$$

as expressed in the geological convention. From here we can further generalize the example and examine the contribution from each face to the $\hat{x}_1$component of the stress vector on the general face, which we have shown is the result of summing the contributions on the rest of the tetrahedron:

$$\boldsymbol{T}_1(\hat{x}_1) = (\boldsymbol{n}_1\sigma_{11} + \boldsymbol{n}_2\sigma_{21} + \boldsymbol{n}_3\sigma_{31})\hat{x}_1$$

In indicial notation, all the cases can be summarized as:

$$\boldsymbol{T}_i = \boldsymbol{n}_j\sigma_{ji}$$

-> Acoustic Wave Equation

where σ_{ji} is the general stress tensor and $\boldsymbol{n}_j$ is the component of the normal to the plane.

Some examples of stress include:

(1) Hydrostatic stress:

$$\begin{pmatrix} \rho gh & 0 & 0 \\ 0 & \rho gh & 0 \\ 0 & 0 & \rho gh \end{pmatrix}$$

,where ρ is the earth's density, g gravitational acceleration and h depth in the earth's crust.

(2) uni-directional stress (say only in the $\hat{x}_1$) superimposed on lithostatic stress

$$\begin{pmatrix} \rho gh + s_{11} & 0 & 0 \\ s_{21} & \rho gh & 0 \\ s_{31} & 0 & \rho gh \end{pmatrix}$$

(3) In many geological cases, below a few kilometers depth, general stress is representable simply as:

+